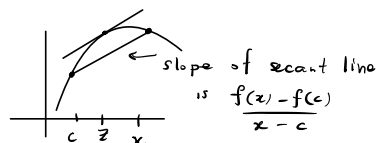


8.7 Taylor's Theorem

Theorem (Mean Value Theorem) Let $f(x)$ be differentiable on an interval I that contains c .

Then for every x in I , there exists z between

$$x \text{ and } c \text{ so that } f'(z) = \frac{f(x) - f(c)}{x - c}$$



$$\Rightarrow f(x) = f(c) + f'(z)(x - c)$$

Taylor's theorem generalizes the Mean Value theorem

Theorem (Taylor's Theorem) Let $f(x)$ be a function that is $(n+1)$ -times differentiable on an interval I that contains c . Then for every x in I ,

$$f(x) = \underbrace{f(c) + f'(c)(x-c) + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n}_{\text{Taylor polynomial } p_n(x) \text{ of degree } n \text{ at } c} + \underbrace{R_n(x)}_{\text{"remainder" or "error"}}$$

$$\text{and } |R_n(x)| \leq \frac{M}{(n+1)!} |x-c|^{n+1} \text{ where}$$

M is the maximum value of $|f^{(n+1)}(z)|$

over all z between x and c .

Summary A function is equal to its Taylor polynomial of degree n , up to an error that can be approximated using the $(n+1)$ st derivative of the function.

Example Let $f(x) = \cos x$ and let $p_6(x)$ be the Maclaurin polynomial of f of degree 6.

- ① Estimate $|f(2) - p_6(2)|$
- ② Find n so that $|f(2) - p_n(2)| < 0.001$

$$\textcircled{1} \quad p_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6$$

$$\begin{aligned} |f(2) - p_6(2)| &= |R_6(2)| \\ &\leq \frac{M}{7!} |2-0|^7 \end{aligned}$$

And M is the maximum of $|f^{(7)}(z)|$ over all z between 0 and 2. Note

$$|f^{(7)}(z)| = |\sin(z)| \leq 1, \text{ so } M \leq 1 \text{ and}$$

$$|f(2) - p_6(2)| \leq \frac{2^7}{7!} \approx 0.02340$$

② We want n so that

$$|f(2) - p_n(2)| = |R_n(2)| \leq 0.001$$

$$\text{Since } |R_n(2)| \leq \frac{M}{(n+1)!} 2^{n+1} \leq \frac{2^{n+1}}{(n+1)!}$$

It will be sufficient to find n so that

$$\frac{2^{n+1}}{(n+1)!} \leq 0.001$$

which we can do by trial and error:

$$\underline{n=8} \quad \frac{2^{8+1}}{(8+1)!} \approx 0.0014, \quad \underline{n=9} \quad \frac{2^{9+1}}{(9+1)!} \approx 0.000282$$

So $n=9$ works (though $p_9(x) = p_8(x)$,
so we can use $p_8(x)$)

Example Let $f(x) = \sqrt{x}$. Use the
degree 4 Taylor polynomial centered at 4
to approximate $\sqrt{3}$ and estimate the error
in this approximation.

$$f'(x) = \frac{1}{2} x^{-1/2}$$

$$f''(x) = -\frac{1}{4} x^{-3/2}$$

$$f'''(x) = \frac{3}{8} x^{-5/2}$$

$$f^{(4)}(x) = -\frac{15}{16} x^{-7/2}$$

$$f(4) = 2$$

$$f'(4) = \frac{1}{2} (2)^{-1} = \frac{1}{4}$$

$$f''(4) = -\frac{1}{4} (2)^{-3} = -\frac{1}{32}$$

$$f'''(4) = \frac{3}{8} (2)^{-5} = \frac{3}{256}$$

$$f^{(4)}(4) = -\frac{15}{16} (2)^{-7} = -\frac{15}{2048}$$

$$P_4(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{32}(x-4)^2 + \frac{3}{256}(x-4)^3 - \frac{15}{2048}(x-4)^4$$

$$P_4(3) = 1.73212$$



$$|f(3) - P_4(3)| = |R_4(3)|$$

$$\leq \frac{M}{5!} |3-4|^5$$

$$= \frac{M}{5!}$$

Note $f^{(5)}(x) = \frac{105}{32} x^{-9/2}$

$$\text{and } M \leq \frac{105}{32} (3)^{-9/2} \leq \frac{105}{32} (1)^{-9/2} = \frac{105}{32}$$

(note $3^{-9/2}$ cannot be computed without knowing $\sqrt{3}$, so we use $1^{-9/2}$ instead)

$$\text{Thus } |f(3) - P_4(3)| \leq \frac{105}{5!} \approx 0.02734$$

Problem 1. For each of the following, approximate the function value with the indicated Taylor polynomial and then give an estimate for the error of the approximation.

- a. Approximate $\sin(0.1)$ using the Maclaurin polynomial of degree 3
 b. Approximate $\sqrt{10}$ using the Taylor polynomial of degree 2 centered at $x=9$

$$(a) \quad P_3(x) = x - \frac{x^3}{3!} \quad P_3(0.1) = 0.1 - \frac{(0.1)^3}{3!} \approx 0.09983$$

$$|P_3(0.1) - \sin(0.1)| = |R_3(0.1)| \\ \leq \frac{M}{4!} |0.1|^4$$

and M is the max of $|f^{(4)}(z)|$ over all z between 0 and 0.1, but $|f^{(4)}(z)| = |\sin z| \leq 1$,
 so $M \leq 1$. So error $\leq \frac{(0.1)^4}{24} = 0.00004167$

$$(b) \quad f(x) = \sqrt{x}, \quad f'(x) = \frac{1}{2}x^{-1/2}, \quad f''(x) = -\frac{1}{4}x^{-3/2}$$

$$P_2(x) = f(9) + f'(9)(x-9) + \frac{f''(9)}{2}(x-9)^2 \\ = 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2$$

$$P_2(10) = 3 + \frac{1}{6} - \frac{1}{216} \approx 3.162037$$

$$|P_2(10) - f(10)| = |R_2(10)| \leq \frac{M}{3!} (10-9)^3 = \frac{M}{3!}$$

where M is max of $|f'''(z)|$ over all z between 9 and 10. $f'''(x) = \frac{3}{8}x^{-5/2}$, so

$$f'''(z) \leq \frac{3}{8}(9)^{-5/2} = \frac{3}{8}(3)^{-5} = \frac{3^{-4}}{8} = \frac{1}{648}$$

since $f'''(z)$ is maximized when $z=9$ (it's decreasing)

$$\text{Thus error} \leq \frac{1}{3!(648)} \approx 0.0002572.$$

Problem 2. Find n so that the Maclaurin polynomial of degree n of $f(x) = e^x$ approximates e within 0.0001 of the actual value.

Need n so that $|R_n(1)| \leq 0.0001$

$$\text{Notice } |R_n(1)| = \frac{M}{(n+1)!} (1-0)^{n+1} = \frac{M}{(n+1)!}$$

where M is the maximum of $|f^{(n+1)}(z)|$ over all z between 0 and 1. Note $|f^{(n+1)}(z)| = e^z$ and $e^z \leq 3$ when z is between 0 and 1 (it's max is $e^1 \leq 3$). Therefore we need

$$\frac{3}{(n+1)!} \leq 0.0001$$

or $(n+1)! \geq 30,000$. $n=7$ works since $8! = 40320$.

Problem 3. Find n so that the Maclaurin polynomial of degree n of $f(x) = \cos x$ approximates $\cos(\pi/3)$ within 0.0001 of the actual value.

Need $|R_n(\pi/3)| \leq 0.0001$. Note

$$|R_n(\pi/3)| = \frac{M}{(n+1)!} \left(\frac{\pi}{3}\right)^{n+1} \text{ where}$$

M is the max of $|f^{(n+1)}(z)|$ over all z between 0 and $\pi/3$. $|f^{(n+1)}(z)| \leq 1$ so need

$$\frac{1}{(n+1)!} \left(\frac{\pi}{3}\right)^{n+1} \leq 0.0001, \text{ and}$$

$n=8$ works.