8.8 Taylor series

Def Let f be a function that is infinitely differentiable at $x=c$ (every derivative is defined at $x=c$). Then the Taylor series of f centered at c is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots$$

When $c=0$, this called the Maclaurin series of f.

Example Give the Maclaurin series of

① $f(x) = e^x$, ② $f(x) = \cos x$

① $f^{(n)}(x) = e^x$ for all $n \geq 0$, so $f^{(n)}(0) = 1$.

Thus $f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \cdots$

$$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3}x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

② $f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x, \ldots$

$f(0) = 1$, $f'(0) = 0$, $f''(0) = -1$, $f'''(0) = 0, \ldots$

Thus the Maclaurin series is

$$1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6}x^6 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$$

Example Give the Taylor series of $f(x) = \frac{1}{x}$ centered at $x = 1$.

$f(x) = \ln x$, $f'(x) = \frac{1}{x}$, $f''(x) = \frac{-1}{x^2}$, $f'''(x) = \frac{2}{x^3}$, $f^{(4)}(x) = \frac{-6}{x^4}$, ...

$f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2$, $f^{(4)}(1) = -6$, ...

Thus the Taylor series is

$$(x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \frac{6}{4!}(x-1)^4 + \dots$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \dots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$$

Common Taylor series and Maclaurin series to know

$\cos x$, $c = 0$, $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} x^{2n}$

$\sin x$, $c = 0$, $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} x^{2n+1}$

e^x, $c = 0$, $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} x^n$

$\frac{1}{1-x}$, $c = 0$, $\displaystyle\sum_{n=0}^{\infty} x^n$

Problem 1. Write out the first four non-zero terms of the Taylor series of f centered at c for the following examples.

a. $f(x) = \sin x, c = \pi/2$

b. $f(x) = \sqrt{x}, c = 4$

c. $f(x) = \ln x, c = 2$

(a) $f(x) = \sin x \qquad f(\pi/2) = 1$

$f'(x) = \cos x \qquad f'(\pi/2) = 0$

$f''(x) = -\sin x \qquad f''(\pi/2) = -1$

$f'''(x) = -\cos x \qquad f'''(\pi/2) = 0$

Taylor series centered at $\pi/2$:

$$1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{6!}\left(x - \frac{\pi}{2}\right)^6 + \dots$$

(b) $f(x) = x^{1/3} \qquad f(8) = 2$

$f'(x) = \frac{1}{3} x^{-2/3} \qquad f'(8) = \frac{1}{3}(2)^{-2} = \frac{1}{12}$

$f''(x) = \frac{-2}{9} x^{-5/3} \qquad f''(8) = \frac{-2}{9}(2)^{-5} = \frac{-1}{144}$

$f'''(x) = \frac{10}{27} x^{-8/3} \qquad f'''(8) = \frac{10}{27}(2)^{-8} = \frac{10}{27} \cdot \frac{1}{256} = \frac{5}{3456}$

Taylor series centered at 8 :

$$2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20736}(x-8)^3 - \dots$$

(c) $f(x) = \ln x \qquad f(2) = \ln 2$

$f'(x) = \frac{1}{x} = x^{-1} \qquad f'(2) = \frac{1}{2}$

$f''(x) = -x^{-2} \qquad f''(2) = -\frac{1}{4}$

$f'''(x) = 2x^{-3} \qquad f'''(2) = \frac{1}{4}$

Taylor series centered at 2 :

$$\ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 - \dots$$

Problem 2. Find the sum of the following series by recognizing they are given by substituting a constant into a known Maclaurin series.

a. $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

b. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} 3^{2n}$

c. $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!}$

d. $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{2^{2n+1}(2n+1)!}$

e. $\sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^{n-1}$

(a) e^2

(b) $= \sum_{n=0}^{\infty} \frac{(-9)^n}{n!} = e^{-9}$

(c) $\cos \pi = -1$

(d) $= \sum_{n=0}^{\infty} (-1)^n \frac{(\pi/2)^{2n+1}}{(2n+1)!} = \sin\left(\frac{\pi}{2}\right) = 1$

(e) $\sum_{n=1}^{\infty} n x^{n-1} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$

$\sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^{n-1} = \frac{1}{(1-\frac{1}{2})^2} = \frac{1}{(\frac{1}{2})^2} = 4$