Theorem Let f be a function that is infinitely differentiable and let R be the radius of convergence of its Taylor series centered at c. Let $R_n(x)$ be the remainder in Taylor's theorem. If $\lim_{n\to\infty} R_n(x) = 0$ for all x in $(c-R, c+R)$ then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

Informal fact All the functions we come across equal their Taylor series on $(c-R, c+R)$. This theorem is stated to emphasize this isn't always the case but that's a subtle issue for more advanced study.

New Taylor series from old ones

We can derive new Taylor series from known ones by doing operations of

1. substitution
2. term-by-term differentiation
3. term-by-term integration

Examples Find the Maclaurin series for the following functions using known ones.

① e^{x^2} ② $\sin x$ ③ $\ln(1+x)$

① $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$

So substituting x^2 in place of x above gives

$$e^{x^2} = 1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^{2n}$$

② $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$

So taking derivative of both sides gives

$$-\sin x = \frac{-2}{2!}x + \frac{4}{4!}x^3 - \frac{6}{6!}x^5 + \cdots$$

$$= -x + \frac{1}{3!}x^3 - \frac{1}{5!}x^5 + \cdots$$

And multiplying both sides by -1, gives

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

③ $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$

So substituting $-x$ into both sides gives

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

and taking integral of both sides gives

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}x^n$$

Using Taylor Series to approximate integrals and compute limits

Example Use term by term integration to approximate $\int_0^1 e^{x^2}\,dx$ use 5 terms of the Maclaurin series of e^{x^2}. this function cannot be integrated by hand!

$$\int_0^1 e^{x^2}\,dx = \int_0^1 \left(1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \frac{1}{4!}x^8 + \cdots\right) dx$$

$$\approx x + \frac{1}{3}x^3 + \frac{1}{5\cdot 2!}x^5 + \frac{1}{7\cdot 3!}x^7 + \frac{1}{8\cdot 4!}x^9 \Big|_0^1$$

$$= 1 + \frac{1}{3} + \frac{1}{10} + \frac{1}{42} + \frac{1}{192}$$

Example Use Maclaurin series to compute $\lim_{x\to 0} \frac{\sin x - x}{x^3}$

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = \lim_{x\to 0} \frac{1}{x^3}\left[\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) - x\right]$$

$$= \lim_{x\to 0} \frac{1}{x^3}\left[-\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]$$

$$= \lim_{x\to 0} \left[\frac{-1}{3!} + \frac{1}{5!}x^2 - \frac{1}{7!}x^4 + \cdots\right]$$

$$= \frac{-1}{3!}$$

Problem 1. Find the Maclaurin series of each function f given below using the given method. Write out the first five non-zero terms and then give the series in summation notation.

a. $f(x) = \frac{1}{1+x^2}$ using substitution into a known Maclaurin series

b. $f(x) = \arctan x$ using term-by-term integration of a known Maclaurin series

c. $f(x) = \frac{1}{(1-x)^2}$ using term-by-term differentiation of a known Maclaurin series

(a)

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + (-x^2)^2 + (-x^2)^3 + (-x^2)^4 + \cdots$$

$$= 1 - x^2 + x^4 - x^6 + x^8 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

(b)

$$\arctan x = \int \frac{1}{1+x^2}\,dx = \int (1 - x^2 + x^4 - x^6 + x^8 - \cdots)\,dx$$

$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

(c)

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\left(1 + x + x^2 + x^3 + x^4 + \cdots\right)$$

$$= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$

$$= \sum_{n=1}^{\infty} n x^{n-1}$$

Problem 2. Approximate the following integrals using the first five non-zero terms of a Maclaurin series.

a. $\int_0^1 \sin(x^2)\,dx$

b. $\int_0^1 x^2 e^{-x^2}\,dx$

(a)

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$$

$$\sin(x^2) = x^2 - \frac{1}{3!}x^6 + \frac{1}{5!}x^{10} - \cdots$$

$$\int_0^1 \sin(x^2)\,dx = \int_0^1 \left(x^2 - \frac{1}{3!}x^6 + \frac{1}{5!}x^{10} - \cdots\right)dx$$

$$\approx \frac{1}{3}x^3 - \frac{1}{42}x^7 + \frac{1}{1320}x^{11}\Big|_0^1 = \frac{1}{3} - \frac{1}{42} + \frac{1}{1320}$$

(b)

$$e^{x} = 1 + x + \frac{1}{2!}x^2 + \cdots$$

$$e^{-x^2} = 1 - x^2 + \frac{1}{2!}x^4 - \cdots$$

$$x^2 e^{-x^2} = x^2 - x^4 + \frac{1}{2!}x^6 - \cdots$$

$$\int_0^1 x^2 e^{-x^2}\,dx = \int_0^1 \left(x^2 - x^4 + \frac{1}{2!}x^6 - \cdots \right) dx$$

$$\approx \frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{1}{14}x^7 \Big|_0^1 = \frac{1}{3} - \frac{1}{5} + \frac{1}{14}$$

Problem 3. Use Maclaurin series to compute the following limits.

a. $\lim_{x \to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}$

b. $\lim_{x \to 0} \frac{x - \ln(1+x)}{x^2}$

(a)

$$\lim_{x \to 0} \frac{\sin x - x + \frac{1}{6}x^6}{x^5} = \lim_{x \to 0} \frac{\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right) - x + \frac{1}{6}x^5}{x^5}$$

$$= \lim_{x \to 0} \frac{\frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots}{x^5}$$

$$= \lim_{x \to 0} \frac{1}{5!} - \frac{1}{7!}x^2 + \cdots$$

$$= \frac{1}{5!} = \frac{1}{120}$$

(b)

$$\lim_{x \to 0} \frac{x - \ln(1+x)}{x^2} = \lim_{x \to 0} \frac{x - \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots\right)}{x^2}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4 - \cdots}{x^2}$$

$$= \lim_{x \to 0} \frac{1}{2} - \frac{1}{3}x + \frac{1}{4}x^2 - \cdots$$

$$= \frac{1}{2}.$$