Problem 1. Use Maclaurin series to evaluate the following limits:

a. $\lim_{x\to 0} \frac{\cos x - 1}{x^2}$

b. $\lim_{x\to 0} \frac{\ln(1-x^2)}{x^2}$

c. $\lim_{x\to 0} \frac{e^{x^2}-x^2-1}{x^4}$

d. $\lim_{x\to 0} \frac{\cos(\sqrt{x})-1}{2x}$

(a)

$$\lim_{x\to 0} \frac{1}{x^2}(\cos x - 1) = \lim_{x\to 0} \frac{1}{x^2}\left[\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots\right) - 1\right]$$

$$= \lim_{x\to 0} \left[-\frac{1}{2!} + \frac{1}{4!}x^2 - \dots\right]$$

$$= -\frac{1}{2}$$

(b)

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

$$-\ln(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots$$

$$\lim_{x\to 0} \frac{1}{x^2}\ln(1-x^2) = \lim_{x\to 0} \frac{-1}{x^2}\left[x^2 + \frac{1}{2}x^4 + \frac{1}{3}x^6 + \dots\right]$$

$$= \lim_{x\to 0} -\left[1 + \frac{1}{2}x^2 + \frac{1}{3}x^4 + \dots\right]$$

$$= -1$$

(c)

$$\lim_{x\to 0} \frac{1}{x^4}\left(e^{x^2} - x^2 - 1\right) = \lim_{x\to 0} \frac{1}{x^4}\left[\left(1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \dots\right) - x^2 - 1\right]$$

$$= \lim_{x\to 0} \frac{1}{2!} + \frac{1}{3!}x^2 + \dots$$

$$= \frac{1}{2!}$$

(d)

$$\lim_{x\to 0} \frac{1}{2x}\left(\cos\sqrt{x} - 1\right) = \lim_{x\to 0} \frac{1}{2x}\left[\left(1 - \frac{1}{2!}x + \frac{1}{4!}x^2 - \frac{1}{6!}x^3 + \dots\right) - 1\right]$$

$$= \lim_{x\to 0} \frac{1}{2}\left[\frac{-1}{2!} + \frac{1}{4!}x - \frac{1}{6!}x^2 + \dots\right]$$

$$= -\frac{1}{4}$$

Problem 2. Find the radius of convergence of the Maclaurin series for $f(x) = 1/(1-2x)$.

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n \quad \text{converges when} \quad |2x| < 1$$

$$\iff |x| < \frac{1}{2}$$

$$\iff R = \frac{1}{2}$$

Problem 3. Suppose $f(0) = 0, f'(0) = 1, f''(0) = -3, f'''(0) = 7, f^{(4)}(0) = -15$. Estimate the value of $\int_0^{0.6} f(x)\,dx$.

$$f(x) \approx x - \frac{3}{2!}x^2 + \frac{7}{3!}x^3 - \frac{15}{4!}x^4$$

$$\Rightarrow \int_0^{0.6} f(x)\,dx \approx \int_0^{0.6} \left(x - \frac{3}{2}x^2 + \frac{7}{6}x^3 - \frac{15}{24}x^4\right) dx$$

$$= \frac{1}{2}x^2 - \frac{1}{2}x^3 + \frac{7}{24}x^4 - \frac{15}{120}x^5 \Big|_0^{0.6}$$

$$= 0.10008$$

Problem 4. Suppose f is infinitely differentiable at 0 and its Maclaurin series is given by

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots.$$

Find $f'(0), f''(0), f'''(0)$, and $f^{(10)}(0)$.

$$f'(0) = 1$$

$$\frac{f''(0)}{2} = \frac{1}{2} \Rightarrow f''(0) = 1$$

$$\frac{f'''(0)}{3!} = \frac{1}{3} \Rightarrow f'''(0) = 2$$

$$\frac{f^{(4)}(0)}{4!} = \frac{1}{4} \Rightarrow f^{(4)}(0) = 3!$$

$$\vdots$$

$$\frac{f^{(10)}(0)}{10!} = \frac{1}{10} \Rightarrow f^{(10)}(0) = 9!$$

Problem 5. Suppose x is a positive but very small number. Arrange the following expressions in increasing order:

$$x, \sin x, \ln(1+x), 1-\cos x, e^x-1, \arctan x.$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots$$

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int (1-x+x^2-x^3+\dots)\,dx = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots$$

$$1-\cos x = 1-\left(1-\frac{1}{2}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\dots\right)$$

$$= \frac{1}{2}x^2 - \frac{1}{4!}x^4 + \frac{1}{6!}x^6 - \dots$$

$$e^x - 1 = x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$$

$$\arctan x = \int \frac{1}{1+x^2}\,dx = \int (1-x^2+x^4-x^6+\dots)\,dx$$

$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$$

$$1-\cos x < \ln(1+x) < \arctan x < \sin x < e^x - 1$$