# §10.4 Taylor's Theorem

**Example** Let $f(x) = \cos x$ and $p_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6$

Let $E_6(x) = f(x) - p_6(x)$. Use a calculator to compute

$E_6(2)$. What could we say about $E_6(2)$ if we didn't

have a calculator that could compute cosine values?

**Theorem** (Taylor's Theorem) Let $f(x)$ be a

function that is $(n+1)$-times differentiable

on an interval $I$ that contains $c$. Then

for every $x$ in $I$,

$$f(x) = \underbrace{f(c) + f'(c)(x-c) + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n}_{\text{Taylor polynomial } p_n(x) \text{ of degree } n \text{ at } c} + \underbrace{E_n(x)}_{\text{"remainder" or "error"}}$$

and $$|E_n(x)| \le \frac{M}{(n+1)!}|x-c|^{n+1}$$ where

$M$ is the maximum value of $|f^{(n+1)}(z)|$

over all $z$ between $x$ and $c$.

**Summary** A function is equal to its Taylor polynomial of degree $n$, up to an error that can be approximated using the $(n+1)$st derivative of the function.

<u>Example</u> Let $f(x) = \cos x$ and let $p_6(x)$ be the Maclaurin polynomial of $f$ of degree 6.

① Estimate $|f(2) - p_6(2)|$

② Find $n$ so that $|f(2) - p_n(2)| < 0.001$

① $p_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6$

$$|f(2) - p_6(2)| = |E_6(2)|$$
$$\leq \frac{M}{7!}|2-0|^7$$

And $M$ is the maximum of $|f^{(7)}(z)|$ over all $z$ between 0 and 2. Note $|f^{(7)}(z)| = |\sin(z)| \leq 1$, so $M \leq 1$ and

$$|f(2) - p_6(2)| \leq \frac{2^7}{7!} \approx 0.02340$$

② We want $n$ so that

$$|f(2) - p_n(2)| = |E_n(2)| \leq 0.001$$

Since $\quad |E_n(2)| \leq \frac{M}{(n+1)!}\,2^{n+1} \leq \frac{2^{n+1}}{(n+1)!}$

It will be sufficient to find $n$ so that

$$\frac{2^{n+1}}{(n+1)!} \leq 0.001$$

which we can do by trial and error:

$\underline{n=8}$ $\quad \frac{2^{8+1}}{(8+1)!} \approx 0.0014$ , $\underline{n=9}$ $\quad \frac{2^{9+1}}{(9+1)!} \approx 0.000282$

So $n=9$ works (though $p_9(x) = p_8(x)$, so we can use $p_8(x)$ )

**Example** Let $f(x) = \sqrt{x}$. Use the degree 4 Taylor polynomial centered at 4 to approximate $\sqrt{3}$ and estimate the error in this approximation.

$f(4) = 2$

$f'(x) = \frac{1}{2} x^{-1/2}$ $\qquad f'(4) = \frac{1}{2}(2)^{-1} = \frac{1}{4}$

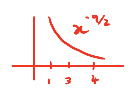
$f''(x) = -\frac{1}{4} x^{-3/2}$ $\qquad f''(4) = -\frac{1}{4}(2)^{-3} = \frac{-1}{32}$

$f'''(x) = \frac{3}{8} x^{-5/2}$ $\qquad f'''(4) = \frac{3}{8}(2)^{-5} = \frac{3}{256}$

$f^{(4)}(x) = \frac{-15}{16} x^{-7/2}$ $\qquad f^{(4)}(4) = \frac{-15}{16}(2)^{-7} = \frac{-15}{2048}$

$$P_4(x) = 2 + \frac{1}{4}(x-4) - \frac{\frac{1}{32}}{2!}(x-4)^2 + \frac{\frac{3}{256}}{3!}(x-4)^3 - \frac{\frac{15}{2048}}{4!}(x-4)^4$$

$P_4(3) = 1.73212$




$$|f(3) - P_4(3)| = |E_4(3)|$$

$$\le \frac{M}{5!}|3-4|^5$$

$$= \frac{M}{5!}$$

Note $f^{(5)}(x) = \frac{105}{32} x^{-9/2}$

and $M \le \frac{105}{32}(3)^{-9/2} \le \frac{105}{32}(1)^{-9/2} = \frac{105}{32}$

(note $3^{-9/2}$ cannot be computed without knowing $\sqrt{3}$, so we use $1^{-9/2}$ instead)

Thus $|f(3) - P_4(3)| \le \frac{\frac{105}{32}}{5!} \approx 0.02734$

**Problem 1.** For each of the following, approximate the function value with the indicated Taylor polynomial and then give an estimate for the error of the approximation.

a. Approximate $\sin(0.1)$ using the Maclaurin polynomial of degree 3

b. Approximate $\sqrt{10}$ using the Taylor polynomial of degree 2 centered at $x = 9$

(a) $P_3(x) = x - \frac{x^3}{3!}$ $\quad P_3(0.1) = 0.1 - \frac{(0.1)^3}{3!} \approx 0.09983$

$$|P_3(0.1) - \sin(0.1)| = |E_3(0.1)| \le \frac{M}{4!}|0.1|^4$$

and $M$ is the max of $|f^{(4)}(z)|$ over all $z$ between 0 and 0.1, but $|f^{(4)}(z)| = |\sin z| \le 1$, so $M \le 1$. So error $\le \frac{(0.1)^4}{24} = 0.000004167$

(b) $f(x) = \sqrt{x}$, $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = \frac{-1}{4}x^{-3/2}$

$$P_2(x) = f(9) + f'(9)(x-9) + \frac{f''(9)}{2}(x-9)^2$$

$$= 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2$$

$$P_2(10) = 3 + \frac{1}{6} - \frac{1}{216} \simeq 3.162037$$

$$|P_2(10) - f(10)| = |E_2(10)| \le \frac{M}{3!}(10-9)^3 = \frac{M}{3!}$$

where $M$ is max of $|f'''(z)|$ over all $z$ between 9 and 10. $f'''(x) = \frac{3}{8}x^{-5/2}$, so

$$f'''(z) \le \frac{3}{8}(9)^{-5/2} = \frac{3}{8}(3)^{-5} = \frac{3^{-4}}{8} = \frac{1}{648}$$

since $f'''(z)$ is maximized when $z = 9$ (it's decreasing)

Thus error $\le \frac{1}{3!(648)} \approx 0.0002572.$

**Problem 2.** Find $n$ so that the Maclaurin polyomial of degree $n$ of $f(x) = e^x$ approximates $e$ within 0.0001 of the actual value.

Need $n$ so that $|R_n(1)| \leq 0.0001$

Notice $|E_n(1)| = \frac{M}{(n+1)!}(1-0)^{n+1} = \frac{M}{(n+1)!}$

where $M$ is the maximum of $|f^{(n+1)}(z)|$ over all $z$ between $0$ and $1$. Note $|f^{(n+1)}(z)| = e^z$ and $e^z \leq 3$ when $z$ is between $0$ and $1$ (it's max is $e^1 \leq 3$). Therefore we need

$$\frac{3}{(n+1)!} \leq 0.0001$$

or $(n+1)! \geq 30,000$. $n = 7$ works since $8! = 40320$.

**Problem 3.** Find $n$ so that the Maclaurin polyomial of degree $n$ of $f(x) = \cos x$ approximates $\cos(\pi/3)$ within 0.0001 of the actual value.

Need $|E_n(\pi/3)| \leq 0.0001$. Note

$$|E_n(\tfrac{\pi}{3})| = \frac{M}{(n+1)!}\left(\frac{\pi}{3}\right)^{n+1}$$ where

$M$ is the max of $|f^{(n+1)}(z)|$ over all $z$ between $0$ and $\pi/3$. $|f^{(n+1)}(z)| \leq 1$ so need

$$\frac{1}{(n+1)!}\left(\pi/3\right)^{n+1} \leq 0.0001,$$ and

$n = 8$ works.