Expected Value

Consider the simple gambling game that requires you to pay \$4 to play and consists one roll of a die.

If you roll a 4, 5, or 6 you win \$8, 10, or 12

If you roll a 1, 2, or 3 you lose \$3, 4, or 5.

Let X be the possible net winnings in one round. What values can X be?

What are your "expected" net winnings?

X can be

$$-7, -8, -9, 4, 6, 8$$

if you roll 1   if you roll 6

expected value of X

= weighted average of possible values of X.

$$= -7\left(\tfrac{1}{6}\right) - 8\left(\tfrac{1}{6}\right) - 9\left(\tfrac{1}{6}\right) + 4\left(\tfrac{1}{6}\right) + 6\left(\tfrac{1}{6}\right) + 8\left(\tfrac{1}{6}\right)$$

$$= \tfrac{1}{6}\left(-24 + 18\right) = -1$$ ("expected" loss of \$1)

Geometric Distribution

Consider the following random experiment:

toss a biased coin with heads probability $1/3$ (and tails prob. $2/3$) repeatedly until you get heads and count how many times you tossed, calling that quantity X.

<u>Exercise</u> What values could X be?

What are the probabilities of getting those values?

X could be $1, 2, 3, \ldots$ (any positive integer)

$$P(X=1) = 1/3$$

$$P(X=2) = \left(\tfrac{2}{3}\right)\left(\tfrac{1}{3}\right)$$

$$P(X=3) = \left(\tfrac{2}{3}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{1}{3}\right) = \left(\tfrac{2}{3}\right)^2\left(\tfrac{1}{3}\right)$$

$$P(X=4) = \underbrace{\left(\tfrac{2}{3}\right)^3}_{\text{3 tails}}\underbrace{\left(\tfrac{1}{3}\right)}_{\text{then heads}}$$

$$P(X=k) = \left(\tfrac{2}{3}\right)^{k-1}\left(\tfrac{1}{3}\right)$$

Exercise What is $\sum_{k=1}^{\infty} P(X = k)$?

(what do you think it "should" be? can you verify?)

$$\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^{k-1}\left(\frac{1}{3}\right) = \frac{1}{3}\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right)$$

$$= \frac{1/3}{1 - 2/3} = 1$$

Exercise What is the "expected" value of X (ie weighted average of possible values)?

$$\sum_{k=1}^{\infty} k\, P(X = k)$$

$$= \sum_{k=1}^{\infty} k \left(\frac{2}{3}\right)^{k-1}\left(\frac{1}{3}\right)$$

$$= \frac{1}{3} \sum_{k=1}^{\infty} k \left(\frac{2}{3}\right)^{k-1}$$

$$= \frac{1}{3} \cdot \frac{1}{\left(1 - \frac{2}{3}\right)^2} = \frac{1}{3} \cdot \frac{1}{(1/3)^2} = 3.$$