Problem 1. Find the interval of convergence of the following power series.

a. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{n^2 5^n}$

b. $\displaystyle\sum_{n=1}^{\infty}\frac{(x+2)^n}{n4^n}$

c. $\displaystyle\sum_{n=1}^{\infty}\frac{2^n(x-2)^n}{(n+2)!}$

a)
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^2 5^{n+1}}\cdot\frac{n^2 5^n}{x^n}\right| = \frac{|x|}{5}$$

Note $\frac{|x|}{5} < 1 \iff |x| < 5$

$\iff -5 < x < 5$

Consider $x = -5$:
$$\sum_{n=1}^{\infty}(-1)^n\frac{(-5)^n}{n^2 5^n} = \sum_{n=1}^{\infty}(-1)^n\frac{(-1)^n 5^n}{n^2 5^n}$$
$$= \sum_{n=1}^{\infty}\frac{1}{n^2}$$ which is

a convergent p-series.

Consider $x = 5$:
$$\sum_{n=1}^{\infty}(-1)^n\frac{5^n}{n^2 5^n} = \sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}$$ which

converges by absolute convergence test

(or alternating series test)

Interval of convergence: $[-5, 5]$.

b) $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{(n+1)4^{n+1}}\cdot\frac{n4^n}{(x+2)^n}\right| = \frac{|x+2|}{4}$$

Note $\frac{|x+2|}{4} < 1 \iff |x+2| < 4$

$\iff -4 < x+2 < 4 \iff -6 < x < 2$

Consider $x=-6$:

$$\sum_{n=1}^{\infty}\frac{(-4)^n}{n4^n} = \sum_{n=1}^{\infty}(-1)^n\frac{4^n}{n4^n}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$ which is

a convergent by alternating series test.

Consider $x=2$:

$$\sum_{n=1}^{\infty}\frac{4^n}{n4^n} = \sum_{n=1}^{\infty}\frac{1}{n}$$ which

is a divergent p-series

Interval of convergence: $[-6, 2)$.

c) $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(x-2)^{n+1}}{(n+3)!}\,\frac{(n+2)!}{2^n(x-2)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2(x-2)}{n+3}\right| = 0 < 1$$

for all x in $(-\infty, \infty)$.

Interval of convergence: $(-\infty, \infty)$

Problem 2. Find the Maclaurin series each of the following functions.

a. $f(x) = \sin(x^4)$

b. $f(x) = xe^{2x}$

c. $f(x) = \ln(1 - x^3)$

d. $f(x) = \dfrac{x^3}{(1+x)^2}$

e. $f(x) = x^4 + 4x^3 + 5x^2 - 3x - 7$

(a)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^7}{9!} - \dots$$

$$\sin(x^4) = x^4 - \frac{x^{12}}{3!} + \frac{x^{20}}{5!} - \frac{x^{28}}{7!} + \frac{x^{36}}{9!} - \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4(2n+1)}$$

(b)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$xe^{2x} = x\left(1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \dots\right)$$

$$= x + 2x^2 + \frac{2^2 x^3}{2!} + \frac{2^3 x^4}{3!} + \frac{2^4 x^5}{4!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{n!} x^{n+1}$$

(c)

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

$$\ln(1-x) = \int \frac{-1}{1-x}\,dx = \int -(1 + x + x^2 + x^3 + x^4 + \dots)\,dx$$

$$= -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{5}x^5 - \dots$$

$$\ln(1-x^3) = -x^3 - \tfrac{1}{2}x^6 - \tfrac{1}{3}x^9 - \tfrac{1}{4}x^{12} - \tfrac{1}{5}x^{15} - \dots$$

$$= \sum_{n=1}^{\infty} \frac{-1}{n} x^{3n}$$

(d) $$\frac{x^3}{(1+x)^2} = -x^3 \frac{d}{dx}\left(\frac{1}{1+x}\right)$$

$$= -x^3 \frac{d}{dx}\left(1 - x + x^2 - x^3 + x^4 - \dots\right)$$

$$= -x^3\left(-1 + 2x - 3x^2 + 4x^3 - 5x^4 + \dots\right)$$

$$= x^3 - 2x^4 + 3x^5 - 4x^6 + 5x^7 - \dots$$

$$= \sum_{n=1}^{\infty} n(-1)^{n+1} x^{n+2}$$

(e) f(x) = its own Taylor series since it's a polynomial

Problem 3. Find the sum of the series below.

a. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!4^n}$

b. $\sum_{n=0}^{\infty} (-1/4)^n$

c. $\sum_{n=1}^{\infty} n(-1/4)^{n-1}$

d. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{4^{2n+1}(2n+1)!}$

e. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{9^n(2n)!}$

(a) $$\sum_{n=0}^{\infty} \frac{(-1/4)^n}{n!} = e^{-1/4}$$

(b) $$\sum_{n=0}^{\infty} (-1/4)^n = \frac{1}{1-(-1/4)} = \frac{4}{5}$$

(c) $$\sum_{n=1}^{\infty} n\left(-1/4\right)^{n-1} = \frac{1}{(1-(-1/4))^2} = \left(\frac{4}{5}\right)^2 = \frac{16}{25}$$

(d) $$\sum_{n=0}^{\infty} (-1)^n \frac{(\pi/4)^{2n+1}}{(2n+1)!} = \sin(\pi/4) = \frac{\sqrt{2}}{2}$$

(e) $$\sum_{n=0}^{\infty} (-1)^n \frac{(\pi/3)^{2n}}{(2n)!} = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$$

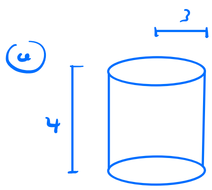
Problem 4. A variety of tanks filled with oil, which has density 800 kg per cubic meter, are given below. Each tank is only filled to half the tank's height. Set up but do not compute an integral for the work performed in pumping all the oil to a height 3 meters above the top of the tank.

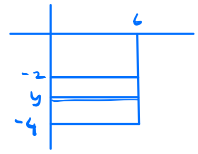
a. An upright circular cylinder with height 4 m and base radius 3 m.

b. A cone whose base is a circle of radius 3 m and whose height is 10 m.

c. A pyramid whose base is a square with side length 4 m and whose height is 7 m.

(a)



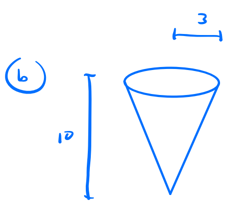


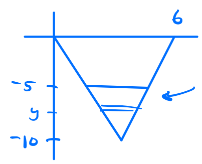
volume of a slice

$= \pi(3)^2\,\Delta y = 9\pi\,\Delta y$

$$\int_{-4}^{-2} \underbrace{(800)(9\pi)}_{\text{mass}}\underbrace{(9.8)}_{g}\underbrace{(3-y)}_{d}\,dy$$

(b)





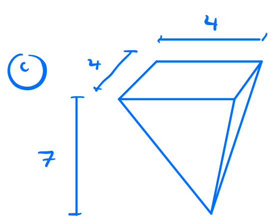
$x = \frac{3}{10}y + 6$

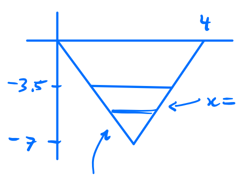
volume of slice

$= \pi\left(\frac{3}{10}y + 3\right)^2 \Delta y$

$$\int_{-10}^{-5} 800\pi\left(\frac{3}{10}y+3\right)^2 (9.8)(3-y)\,dy$$

(c)





$x = \frac{2}{7}y + 4$

$x = -\frac{2}{7}y$

volume of slice

$= \left(\frac{4}{7}y + 4\right)^2 \Delta y$

$$\int_{-7}^{-3.5} 800\left(\frac{4}{7}y+4\right)^2 (9.8)(3-y)\,dy$$

Problem 5. Use Taylor's theorem to find a bound for the error in approximating the given quantity with a third degree Maclaurin polynomial for the given function.

a. $\sin(0.2)$, $f(x) = \sin x$

b. $\sqrt{0.9}$, $f(x) = \sqrt{1+x}$

c. $1/\sqrt{3}$, $f(x) = (1+x)^{-1/2}$

(a) $|R_3(0.2)| = \frac{|f^{(4)}(z)|}{4!} |0.2-0|^4$ for some z between 0 and 0.2

$= \frac{\sin(z)}{4!} (0.2)^4$

$\le \frac{\sin(0.2)}{4!} (0.2)^4$

$\le \frac{1}{4!} (0.2)^4$

(b) $f'(x) = \frac{1}{2}(1+x)^{-1/2}$ $\quad f'''(x) = \frac{3}{8}(1+x)^{-5/2}$

$f''(x) = -\frac{1}{4}(1+x)^{-3/2}$ $\quad f^{(4)}(x) = \frac{-15}{16}(1+x)^{-7/2}$

$|R_4(-0.1)| = \frac{|f^{(4)}(z)|}{4!} (-0.1-0)^4$ for some z between -0.1 and 0

$= \frac{\frac{15}{16}(1+z)^{-7/2}}{4!} (0.1)^4$

$\le \frac{(\frac{15}{16})(.81)^{-7/2}}{4!} (0.1)^4$

(max of $(1+z)^{-7/2}$ on $[-0.1, 0]$ occurs at $z = -0.1$, but $.9^{-7/2}$ cannot be computed, though $0.81^{-7/2}$ can be computed)

(c) $f'(x) = -\frac{1}{2}(1+x)^{-3/2}$ $\quad f'''(x) = \frac{15}{8}(1+x)^{-7/2}$

$f''(x) = \frac{3}{4}(1+x)^{-5/2}$ $\quad f^{(4)}(x) = \frac{-105}{16}(1+x)^{-9/2}$

$|R_4(2)| = \frac{|f^{(4)}(z)|}{4!} |2-0|^4$ for some z between 0 and 2

$= \frac{\frac{105}{16}(1+z)^{-9/2}}{4!} 2^4$

$\le \frac{\frac{105}{16}}{4!} 2^4$

Problem 6. Find the Taylor polynomial of degree 4 of $f(x)$ centered at c for the given examples below.

a. $f(x) = 1/(1+x), c = 2$

b. $f(x) = \sin x, c = -\pi/4$

c. $f(x) = \ln(x^2), c = 1$

(a)

$f(x) = (1+x)^{-1}$ $\quad f(2) = 3^{-1}$

$f'(x) = -(1+x)^{-2}$ $\quad f'(2) = -3^{-2}$

$f''(x) = 2(1+x)^{-3}$ $\quad f''(2) = 2(3)^{-3}$

$f'''(x) = -6(1+x)^{-4}$ $\quad f'''(2) = -6(3)^{-4}$

$f^{(4)}(x) = 24(1+x)^{-5}$ $\quad f^{(4)}(2) = 24(3)^{-5}$

$$P_4(x) = 3^{-1} - 3^{-2}(x-2) + 3^{-3}(x-2)^2 - 3^{-4}(x-2)^3 + 3^{-5}(x-2)^4$$

(b)

$f(x) = \sin x$ $\quad f(-\pi/4) = \frac{-\sqrt{2}}{2}$

$f'(x) = \cos x$ $\quad f'(-\pi/4) = \frac{\sqrt{2}}{2}$

$f''(x) = -\sin x$ $\quad f''(-\pi/4) = \frac{\sqrt{2}}{2}$

$f'''(x) = -\cos x$ $\quad f'''(-\pi/4) = -\frac{\sqrt{2}}{2}$

$f^{(4)}(x) = \sin x$ $\quad f^{(4)}(-\pi/4) = -\frac{\sqrt{2}}{2}$

$$P_4(x) = \frac{-\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x+\frac{\pi}{4}\right) + \frac{\sqrt{2}}{2\cdot 2!}\left(x+\frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{2\cdot 3!}\left(x+\frac{\pi}{4}\right)^3 - \frac{\sqrt{2}}{2\cdot 4!}\left(x+\frac{\pi}{4}\right)^4$$

(c) $f(x) = \ln(x^2) = 2\ln x$ $\quad f(1) = 0$

$f'(x) = 2x^{-1}$ $\quad f'(1) = 2$

$f''(x) = -2x^{-2}$ $\quad f''(1) = -2$

$f'''(x) = 4x^{-3}$ $\quad f'''(1) = 4$

$f^{(4)}(x) = -12x^{-4}$ $\quad f^{(4)}(1) = -12$

$$P_4(x) = 2(x-1) - (x-1)^2 + \frac{2}{3}(x-1)^3 - \frac{1}{2}(x-1)^4$$