Problem 1. Find the interval of convergence of the following power series.

a. $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{n^2 5^n}$

b. $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n4^n}$

c. $\sum_{n=1}^{\infty} \frac{2^n(x-2)^n}{(n+2)!}$

a)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^2 5^{n+1}}\cdot\frac{n^2 5^n}{x^n}\right| = \frac{|x|}{5}$$

Note $\frac{|x|}{5} < 1 \iff |x| < 5$

$\iff -5 < x < 5$

Consider $x = -5$:

$$\sum_{n=1}^{\infty}(-1)^n \frac{(-5)^n}{n^2 5^n} = \sum_{n=1}^{\infty}(-1)^n \frac{(-1)^n 5^n}{n^2 5^n}$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^2}$$ which is

a convergent p-series.

Consider $x = 5$:

$$\sum_{n=1}^{\infty}(-1)^n \frac{5^n}{n^2 5^n} = \sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}$$ which

converges by absolute convergence test

(or alternating series test)

Interval of convergence: $[-5, 5]$.

b) $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+2)^{n+1}}{(n+1)4^{n+1}}\cdot\frac{n4^n}{(x+2)^n}\right| = \frac{|x+2|}{4}$$

Note $\frac{|x+2|}{4} < 1 \iff |x+2| < 4$

$\iff -4 < x+2 < 4 \iff -6 < x < 2$

Consider $\underline{x=-6}$:

$$\sum_{n=1}^{\infty}\frac{(-4)^n}{n4^n} = \sum_{n=1}^{\infty}(-1)^n\frac{4^n}{n4^n}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$ which is

a convergent by alternating series test.

Consider $\underline{x=2}$:

$$\sum_{n=1}^{\infty}\frac{4^n}{n4^n} = \sum_{n=1}^{\infty}\frac{1}{n}$$ which

is a divergent p-series

Interval of convergence: $[-6, 2)$.

c) $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(x-2)^{n+1}}{(n+3)!}\cdot\frac{(n+2)!}{2^n(x-2)^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2(x-2)}{n+3}\right| = 0 < 1$$

for all x in $(-\infty,\infty)$.

Interval of convergence: $(-\infty,\infty)$

Problem 2. Find the first 5 terms of the Taylor series about 0 of each of the following functions and then write the full series using summation notation.

a. $f(x) = \sin(x^4)$

b. $f(x) = xe^{2x}$

c. $f(x) = \ln(1 - x^3)$

d. $f(x) = \dfrac{x^3}{(1+x)^2}$

e. $f(x) = x^4 + 4x^3 + 5x^2 - 3x - 7$

(a)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^7}{9!} - \dots$$

$$\sin(x^4) = x^4 - \frac{x^{12}}{3!} + \frac{x^{20}}{5!} - \frac{x^{28}}{7!} + \frac{x^{36}}{9!} - \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4(2n+1)}$$

(b)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$xe^{2x} = x\left(1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \dots\right)$$

$$= x + 2x^2 + \frac{2^2x^3}{2!} + \frac{2^3x^4}{3!} + \frac{2^4x^5}{4!} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{2^n}{n!} x^{n+1}$$

(c)

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

$$\ln(1-x) = \int \frac{-1}{1-x}\,dx = \int -(1 + x + x^2 + x^3 + x^4 + \dots)\,dx$$

$$= -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{5}x^5 - \dots$$

$$\ln(1-x^3) = -x^3 - \frac{1}{2}x^6 - \frac{1}{3}x^9 - \frac{1}{4}x^{12} - \frac{1}{5}x^{15} - \dots$$

$$= \sum_{n=1}^{\infty} \frac{-1}{n} x^{3n}$$

(d)

$$\frac{x^3}{(1+x)^2} = -x^3 \frac{d}{dx}\left(\frac{1}{1+x}\right)$$

$$= -x^3 \frac{d}{dx}\left(1 - x + x^2 - x^3 + x^4 - \dots\right)$$

$$= -x^3\left(-1 + 2x - 3x^2 + 4x^3 - 5x^4 + \dots\right)$$

$$= x^3 - 2x^4 + 3x^5 - 4x^6 + 5x^7 - \dots$$

$$= \sum_{n=1}^{\infty} n(-1)^{n+1} x^{n+2}$$

(e) $f(x) =$ its own Taylor series since it's a polynomial

Problem 3. Find the sum of the series below using known Taylor series.

a. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!4^n}$

b. $\sum_{n=0}^{\infty} (-1/4)^n$

c. $\sum_{n=1}^{\infty} n(-1/4)^{n-1}$

d. $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{4^{2n+1}(2n+1)!}$

(a) $$\sum_{n=0}^{\infty} \frac{(-1/4)^n}{n!} = e^{-1/4}$$

(b) $$\sum_{n=0}^{\infty} (-1/4)^n = \frac{1}{1-(-1/4)} = \frac{4}{5}$$

(c) $$\sum_{n=1}^{\infty} n\left(-1/4\right)^{n-1} = \frac{1}{(1-(-1/4))^2} = \left(\frac{4}{5}\right)^2 = \frac{16}{25}$$

(d) $$\sum_{n=0}^{\infty} (-1)^n \frac{(\pi/4)^{2n+1}}{(2n+1)!} = \sin(\pi/4) = \frac{\sqrt{2}}{2}$$

Problem 4. Use Taylor's theorem to find a bound for the error in approximating the given quantity with a third degree Taylor polynomial about 0 for the given function.

a. $\sin(0.2)$, $f(x) = \sin x$

b. $\sqrt{0.9}$, $f(x) = \sqrt{1+x}$

c. $1/\sqrt{3}$, $f(x) = (1+x)^{-1/2}$

(a) $|E_3(0.2)| \le \frac{M}{4!}|0.2-0|^4$ where M is the max of $|f^{(4)}(z)|$ for z between 0 and 0.2

$= \frac{\sin(0.2)}{4!}(0.2)^4$

$< \frac{1}{4!}(0.2)^4$ since $\sin(0.2) < \sin\left(\frac{\pi}{2}\right) = 1$

(b) $f'(x) = \frac{1}{2}(1+x)^{-1/2}$ $\quad f'''(x) = \frac{3}{8}(1+x)^{-5/2}$

$f''(x) = -\frac{1}{4}(1+x)^{-3/2}$ $\quad f^{(4)}(x) = \frac{-15}{16}(1+x)^{-7/2}$

$|E_4(-0.1)| \le \frac{M}{4!}(-0.1-0)^4$ where M is the max of $|f^{(4)}(z)|$ for z between -0.1 and 0

$\le \frac{\left(\frac{15}{16}\right)(.81)^{-7/2}}{4!}(0.1)^4$

(max of $(1+z)^{-7/2}$ on $[-0.1,0]$ occurs at $z=-0.1$, but $.9^{-7/2}$ cannot be computed, though $0.81^{-7/2}$ can be computed)

(c) $f'(x) = -\frac{1}{2}(1+x)^{-3/2}$ $\quad f'''(x) = \frac{15}{8}(1+x)^{-7/2}$

$f''(x) = \frac{3}{4}(1+x)^{-5/2}$ $\quad f^{(4)}(x) = \frac{-105}{16}(1+x)^{-9/2}$

$|E_4(2)| \le \frac{M}{4!}|2-0|^4$ where M is the max of $|f^{(4)}(z)|$ for z between 0 and 2

$= \frac{\frac{105}{16}}{4!}2^4$

(max occurs at 0)

Problem 5. Find the Taylor polynomial of degree 4 of $f(x)$ about c for the given examples below.

a. $f(x) = 1/(1+x), c = 2$

b. $f(x) = \sin x, c = -\pi/4$

c. $f(x) = \ln(x^2), c = 1$

(a)
$$f(x) = (1+x)^{-1} \qquad f(2) = 3^{-1}$$
$$f'(x) = -(1+x)^{-2} \qquad f'(2) = -3^{-2}$$
$$f''(x) = 2(1+x)^{-3} \qquad f''(2) = 2(3)^{-3}$$
$$f'''(x) = -6(1+x)^{-4} \qquad f'''(2) = -6(3)^{-4}$$
$$f^{(4)}(x) = 24(1+x)^{-5} \qquad f^{(4)}(2) = 24(3)^{-5}$$

$$P_4(x) = 3^{-1} - 3^{-2}(x-2) + 3^{-3}(x-2)^2 - 3^{-4}(x-2)^3 + 3^{-5}(x-2)^4$$

(b)
$$f(x) = \sin x \qquad f(-\pi/4) = \frac{-\sqrt{2}}{2}$$
$$f'(x) = \cos x \qquad f'(-\pi/4) = \frac{\sqrt{2}}{2}$$
$$f''(x) = -\sin x \qquad f''(-\pi/4) = \frac{\sqrt{2}}{2}$$
$$f'''(x) = -\cos x \qquad f'''(-\pi/4) = -\frac{\sqrt{2}}{2}$$
$$f^{(4)}(x) = \sin x \qquad f^{(4)}(-\pi/4) = -\frac{\sqrt{2}}{2}$$

$$P_4(x) = \frac{-\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x+\frac{\pi}{4}\right) + \frac{\sqrt{2}}{2\cdot 2!}\left(x+\frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{2\cdot 3!}\left(x+\frac{\pi}{4}\right)^3 - \frac{\sqrt{2}}{2\cdot 4!}\left(x+\frac{\pi}{4}\right)^4$$

(c) $f(x) = \ln(x^2) = 2\ln x$ $\quad f(1) = 0$

$f'(x) = 2x^{-1}$ $\quad f'(1) = 2$

$f''(x) = -2x^{-2}$ $\quad f''(1) = -2$

$f'''(x) = 4x^{-3}$ $\quad f'''(1) = 4$

$f^{(4)}(x) = -12x^{-4}$ $\quad f^{(4)}(1) = -12$

$$P_4(x) = 2(x-1) - (x-1)^2 + \frac{2}{3}(x-1)^3 - \frac{1}{2}(x-1)^4$$