§7.1 Substitution, continued

Today we'll continue our work on substitution but now in the context of definite integrals.

Examples

① $\displaystyle\int_2^3 x^2 e^{x^3}\,dx$

$w = x^3$

$dw = 3x^2\,dx$

$\frac{1}{3}\,dw = x^2\,dx$

$$= \int_8^{27} \frac{1}{3} e^w\,dw$$

when we change variables (ie. substitute w in) we must change our limits of integration:

$x = 2 \implies w = 2^3 = 8$

$x = 3 \implies w = 3^3 = 27$

$$= \frac{1}{3} e^w \Big|_8^{27}$$

$$= \frac{1}{3}\left(e^{27} - e^8\right)$$

notice we did not back-substitute

② $\int_0^2 \cos(3x-1)\,dx$

$w = 3x-1$

$dw = 3\,dx$

$\frac{1}{3}\,dw = dx$

$= \int_{-1}^{5} \frac{1}{3}\cos(w)\,dw$

$= \frac{1}{3}\sin(w)\Big|_{-1}^{5}$

$= \frac{1}{3}\left(\sin(5) - \sin(-1)\right)$

③ $\int_{16}^{25} \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx$

$w = \sqrt{x} = x^{1/2}$

$dw = \frac{1}{2}x^{-\frac{1}{2}}\,dx$

$2\,dw = x^{-\frac{1}{2}}\,dx$

$= \frac{1}{\sqrt{x}}\,dx$

$= 2\int_{4}^{5} \cos(w)\,dw$

$= 2\sin(w)\Big|_{4}^{5}$

$= 2\left(\sin(5) - \sin(4)\right)$

§ 7.2 Integration by parts

In this section, we're going to learn a method for integrating products of functions when substitution is not a viable method.

Example $\int x \cos x \, dx$ substitution won't work!

Basic idea: Let $u(x)$ and $v(x)$ be given functions.

Product rule for derivatives:

$$\frac{d}{dx}\left(u(x) \cdot v(x)\right) = u'(x)\, v(x) + u(x) \cdot v'(x).$$

Integrate both sides with respect to x:

$$\int \frac{d}{dx}(u \cdot v)\, dx = \int u' \cdot v \, dx + \int u \cdot v' \, dx$$

Simplify:

$$u \cdot v = \int u' \cdot v \, dx + \int u \cdot v' \, dx$$

Rearrange:

$$\int u \cdot v' \, dx = u \cdot v - \int u' \cdot v \, dx$$

Shorthand notation:

$$\boxed{\int u \, dv = uv - \int v \, du}$$

This is called the integration by parts formula.

Main challenge: decide what u and dv should be.

Examples

① $\int \underbrace{x}_{u} \underbrace{\cos x \, dx}_{dv}$

$u = x \qquad dv = \cos x \, dx$

$du = dx \qquad v = \sin x$

$= \underbrace{x}_{u} \underbrace{\sin x}_{v} - \int \underbrace{\sin x}_{v} \underbrace{dx}_{du}$

$= x \sin x + \cos x + C.$

② $\int \underbrace{x}_{u} \underbrace{e^{x} \, dx}_{dv}$

$u = x \qquad dv = e^{x} \, dx$

$du = dx \qquad v = e^{x}$

$= \underbrace{x}_{u} \underbrace{e^{x}}_{v} - \int \underbrace{e^{x}}_{v} \underbrace{dx}_{du}$

$= x e^{x} - e^{x} + C$

③ $\int \underbrace{\ln x}_{u} \underbrace{dx}_{dv}$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} \, dx \qquad v = x$

$= \underbrace{x}_{u} \underbrace{\ln x}_{v} - \int \underbrace{x}_{v} \cdot \underbrace{\frac{1}{x} \, dx}_{du}$

$= x \ln x - \int 1 \, dx$

$= x \ln x - x + C$

④ $\int x^5 \ln x \, dx$

$u = \ln x \qquad dv = x^5 \, dx$

$du = \frac{1}{x} dx \qquad v = \frac{1}{6} x^6$

$= \frac{1}{6} x^6 \ln x - \int \frac{1}{6} x^6 \cdot \frac{1}{x} \, dx$

$= \frac{1}{6} x^6 \ln x - \frac{1}{6} \int x^5 \, dx$

$= \frac{1}{6} x^6 \ln x - \frac{1}{36} x^6 + C.$

Rule of Thumb for choosing u

In order, starting from highest priority, choose u to be a

L ogarithmic function

I nverse trig function

P olynomial

E xponential function

T rig function

Problem 1. Find the following definite integrals using substitution. Identify w, compute $dw = w'(x)\,dx$, and convert the limits of integration to start.

a. $\int_0^2 \sqrt{5x+2}\,dx$

b. $\int_0^1 (8x+2)(2x^2+x)^4\,dx$

c. $\int_0^{\pi} \cos\left(\frac{x}{2}+\pi\right)\,dx$

d. $\int_0^{\pi/2} e^{-\cos\theta}\sin\theta\,d\theta$

e. $\int_1^8 \frac{e^{x^{1/3}}}{x^{2/3}}\,dx$

ⓐ $\frac{1}{5} \int_2^{12} \sqrt{w} \, dw$

$w = 5x + 2$

$dw = 5dx$

$\frac{1}{5} dw = dx$

$= \frac{1}{5} \int_2^{12} w^{1/2} \, dw$

$= \frac{2}{15} w^{3/2} \Big|_2^{12} = \frac{2}{15} \left(12^{3/2} - 2^{3/2}\right)$

(b) $\int_0^3 2w^4\,dw$

$w = 2x^2 + x$

$dw = (4x+1)\,dx$

$2\,dw = (8x+2)\,dx$

$= \frac{2}{5} w^5 \Big|_0^3$

$= \frac{2}{5}(243)$

$= \frac{486}{5}$

(c) $\int_{\pi}^{\frac{3\pi}{2}} 2\cos(w)\,dw$

$w = \frac{x}{2} + \pi$

$dw = \frac{1}{2}\,dx$

$2\,dw = dx$

$= 2\sin w \Big|_{\pi}^{\frac{3\pi}{2}}$

$= 2\left(\sin\left(\frac{3\pi}{2}\right) - \sin(\pi)\right) = -2.$

(d) $\int_{-1}^{0} e^w\,dw$

$w = -\cos\theta$

$dw = \sin\theta\,d\theta$

$= e^w \Big|_{-1}^{0}$

$= e^0 - e^{-1}$

$= 1 - e^{-1}$

(e) $\int_1^2 3e^w\,dw$

$w = x^{1/3}$

$dw = \frac{1}{3}x^{-2/3}\,dx$

$3\,dw = x^{-2/3}\,dx$

$= 3e^w \Big|_1^2$

$= 3(e^2 - e)$

Problem 2. Find the following indefinite integrals using integration by parts.

a. $\int x \sin x \, dx$

b. $\int x e^{2x} \, dx$

c. $\int x^4 \ln x \, dx$

d. $\int x^2 \sin x \, dx$

e. $\int \cos^2 x \, dx$

(a) $-x\cos x + \int \cos x \, dx$ $\quad u = x \quad dv = \sin x \, dx$

$du = dx \quad v = -\cos x$

$= -x\cos x + \sin x + C$

(b) $\frac{1}{2} x e^{2x} - \int \frac{1}{2} e^{2x} \, dx$ $\quad u = x \quad dv = e^{2x} \, dx$

$du = dx \quad v = \frac{1}{2} e^{2x}$

$= \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + C$

(c) $\frac{1}{5} x^5 \ln x - \int \frac{1}{5} x^4 \, dx$ $\quad u = \ln x \quad dv = x^4 \, dx$

$du = \frac{1}{x} dx \quad v = \frac{1}{5} x^5$

$= \frac{1}{5} x^5 \ln x - \frac{1}{25} x^5 + C$

(d) $\quad u = x^2 \quad dv = \sin x \, dx$

$-x^2 \cos x + \int 2x \cos x \, dx$ $\quad du = 2x\,dx \quad v = -\cos x$

$= -x^2 \cos x + 2x \sin x - \int 2\sin x \, dx$ $\quad u = 2x \quad dv = \cos x \, dx$

$= -x^2 \cos x + 2x\sin x + 2\cos x + C$ $\quad du = 2\,dx \quad v = \sin x$

$u = \cos x \quad dv = \cos x \, dx$

(e) $\int \cos^2 x \, dx = \sin x \cos x + \int \sin^2 x \, dx$ $\quad du = -\sin x \, dx \quad v = \sin x$

$= \sin x \cos x + \int (1 - \cos^2 x) \, dx$

$= \sin x \cos x + \int dx - \int \cos^2 x \, dx$

$= \sin x \cos x + x - \int \cos^2 x \, dx$

Let $I = \int \cos^2 x \, dx$. Then

$I = \sin x \cos x + x - I$.

So $2I = \sin x \cos x + x$, which means $I = \frac{1}{2} \sin x \cos x + \frac{1}{2} x + C$.