§7.2 Integration by parts, continued

Examples

① $\displaystyle\int_0^{\pi/2} x\sin(2x)\,dx$

$u = x \qquad dv = \sin(2x)\,dx$

$du = dx \qquad v = -\frac{1}{2}\cos(2x)$

$$= -\frac{1}{2}x\cos(2x)\Big|_0^{\pi/2} + \int_0^{\pi/2}\frac{1}{2}\cos(2x)\,dx$$

$$= -\frac{1}{2}\left(\frac{\pi}{2}\cos(\pi) - 0\cdot\cos(0)\right) + \frac{1}{2}\sin(2x)\Big|_0^{\pi/2}$$

$$= \frac{\pi}{4}.$$

② $\displaystyle\int x^2\sin x\,dx$

$u = x^2 \qquad dv = \sin x\,dx$

$du = 2x\,dx \qquad v = -\cos x$

$$-x^2\cos x + \int 2x\cos x\,dx$$

$u = 2x \qquad dv = \cos x\,dx$

$du = 2\,dx \qquad v = \sin x$

$$= -x^2\cos x + 2x\sin x - \int 2\sin x\,dx$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + C$$

③ $\int \cos^2 x \, dx$

$$\int \cos^2 x \, dx = \sin x \cos x + \int \sin^2 x \, dx$$

$u = \cos x \qquad dv = \cos x \, dx$

$du = -\sin x \, dx \qquad v = \sin x$

$$= \sin x \cos x + \int (1 - \cos^2 x) \, dx$$

$$= \sin x \cos x + \int dx - \int \cos^2 x \, dx$$

$$= \sin x \cos x + x - \int \cos^2 x \, dx$$

Let $I = \int \cos^2 x \, dx$. Then

$$I = \sin x \cos x + x - I.$$

So $2I = \sin x \cos x + x$, which means $I = \frac{1}{2} \sin x \cos x + \frac{1}{2} x + C$.

④ $\int e^{2x} \sin(3x) \, dx$

$u = e^{2x} \qquad dv = \sin(3x) \, dx$

$du = 2e^{2x} \, dx \qquad v = -\frac{1}{3} \cos(3x)$

$$= -\frac{1}{3} e^{2x} \cos(3x) + \int \frac{2}{3} e^{2x} \cos(3x) \, dx$$

$u = \frac{2}{3} e^{2x} \qquad dv = \cos(3x) \, dx$

$du = \frac{4}{3} e^{2x} \, dx \qquad v = \frac{1}{3} \sin(3x)$

$$= -\frac{1}{3} e^{2x} \cos(3x) + \frac{2}{9} e^{2x} \sin(3x) - \frac{4}{9} \int e^{2x} \sin(3x) \, dx$$

Let $I = \int e^{2x} \sin(3x)\,dx$. Then

$$I = -\frac{1}{3} e^{2x} \cos(3x) + \frac{2}{9} e^{2x} \sin(3x) - \frac{4}{9} I$$

$$\Rightarrow \quad \frac{13}{9} I = -\frac{1}{3} e^{2x} \cos(3x) + \frac{2}{9} e^{2x} \sin(3x)$$

$$\Rightarrow \quad I = \frac{9}{13}\left(-\frac{1}{3} e^{2x} \cos(3x) + \frac{2}{9} e^{2x} \sin(3x)\right) + C$$

$$= -\frac{3}{13} e^{2x} \cos(3x) + \frac{2}{13} e^{2x} \sin(3x) + C.$$

Problem 1. Find the following integrals using integration by parts.

a. $\int_1^9 \ln(2x)\,dx$

b. $\int_0^1 xe^{-x}\,dx$

c. $\int x^2 e^{3x}\,dx$

d. $\int \sin^2 x\,dx$

e. $\int x^3 e^{x^2}\,dx$ (Hint: try using substitution followed by integration by parts)

(a) $x \ln(2x) \Big|_1^9 - \int_1^9 dx$

$u = \ln(2x) \qquad dv = dx$

$du = \frac{1}{x}\,dx \qquad v = x$

$= 9 \ln 18 - \ln 2 - 8$

(b) $-xe^{-x} \Big|_0^1 + \int_0^1 e^{-x}\,dx$

$u = x \qquad dv = e^{-x}\,dx$

$du = dx \qquad v = -e^{-x}$

$= -e^{-1} - e^{-x} \Big|_0^1$

$= -e^{-1} - (e^{-1} - 1)$

$= 1 - 2e^{-1}$

ⓒ $\frac{1}{3}x^2 e^{3x} - \int \frac{2}{3} x e^{3x}\,dx$

$= \frac{1}{3}x^2 e^{3x} - \frac{2}{9} x e^{3x} - \int \frac{2}{9} e^{3x}\,dx$

$= \frac{1}{3}x^2 e^{3x} - \frac{2}{9} x e^{3x} - \frac{2}{27} e^{3x} + C$

$u = x^2 \qquad dv = e^{3x}\,dx$

$du = 2x\,dx \qquad v = \frac{1}{3}e^{3x}$

$u = \frac{2}{3}x \qquad dv = e^{3x}\,dx$

$du = \frac{2}{3}\,dx \qquad v = \frac{1}{3}e^{3x}$

ⓓ $-\sin x \cos x + \int \cos^2 x\,dx$

$= -\sin x \cos x + \int (1 - \sin^2 x)\,dx$

$= -\sin x \cos x + x - \int \sin^2 x\,dx.$

Let $I = \int \sin^2 x\,dx$. Then

$I = -\sin x \cos x + x - I$

$\Rightarrow \quad 2I = -\sin x \cos x + x$

$\Rightarrow \quad I = -\frac{1}{2}\sin x \cos x + \frac{1}{2}x.$

$u = \sin x \qquad dv = \sin x\,dx$

$du = \cos x \qquad v = -\cos x$

ⓔ $\int x^3 e^{x^2}\,dx$

$= \int x \cdot x^2 \cdot e^{x^2}\,dx$

$= \int \frac{1}{2} w e^w\,dw$

$= we^w - \int e^w\,dw$

$= we^w - e^w + C$

$= x^2 e^{x^2} - e^{x^2} + C$

$w = x^2$

$dw = 2x\,dx$

$\frac{1}{2}dw = x\,dx$

$u = w \qquad dv = e^w\,dw$

$du = dw \qquad v = e^w$