8.3 Comparison tests, continued

Last time (p-series test) Consider the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ (called a p-series)

1. When $p \leq 1$, the series diverges
2. When $p > 1$, the series converges

Theorem (Comparison test) Suppose $0 \leq a_n \leq b_n$ for all $n \geq 1$ and consider the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$.

1. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.
2. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Problem 1. For each a_n and b_n pair below, decide which is true:

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n \quad \text{or} \quad \sum_{n=1}^{\infty} a_n \geq \sum_{n=1}^{\infty} b_n.$$

Notice that in each pair, $\sum_{n=1}^{\infty} b_n$ is a series that you know converges or diverges using your established knowledge about p-series and geometric series. Use this to make a conclusion about whether $\sum_{n=1}^{\infty} a_n$ converges or diverges if possible.

a. $a_n = \frac{1}{n^2+4n+7}, b_n = \frac{1}{n^2}$

b. $a_n = \frac{1}{n^{3.5}+n^2+n+4}, b_n = \frac{1}{n^{3.5}}$

c. $a_n = \frac{n^2+n+1}{n^3}, b_n = \frac{1}{n}$

d. $a_n = \frac{1}{\sqrt{n^2-0.5}}, b_n = \frac{1}{n}$

e. $a_n = \frac{2^n}{5^n+10}, b_n = \frac{2^n}{5^n}$

(b) $a_n \leq b_n$, $\sum_{n=1}^{\infty} b_n$ converges so $\sum_{n=1}^{\infty} a_n$ converges

(c) $a_n \geq b_n$, $\sum_{n=1}^{\infty} b_n$ diverges, so $\sum_{n=1}^{\infty} a_n$ diverges

(d) $a_n \geq b_n$, $\sum_{n=1}^{\infty} b_n$ diverges, so $\sum_{n=1}^{\infty} a_n$ diverges

(e) $a_n \leq b_n$, $\sum_{n=1}^{\infty} b_n$ converges, so $\sum_{n=1}^{\infty} a_n$ converges.

Example Use the comparison test to explain the convergence / divergence of the series:

① $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^3+3}$

② $\displaystyle\sum_{n=1}^{\infty} \frac{6n^2+1}{2n^3-1}$

① Note $n-1 \leq n$

and $n^3+3 \geq n^3$ for all n.

Then $\dfrac{1}{n^3+1} \leq \dfrac{n}{n^3} = \dfrac{1}{n^2}$ for all n.

By comp. test, $\sum \dfrac{n-1}{n^3+3}$ converges

since $\sum \dfrac{1}{n^2}$ converges by p-series test

② Note $6n^2+1 \geq 6n^2$

and $2n^3-1 \leq 2n^3$ for all n.

Then $\dfrac{6n^2+1}{2n^3-1} \geq \dfrac{6n^2}{2n^3} = \dfrac{3}{n}$ for all n.

By comp. test $\sum \dfrac{6n^2+1}{2n^3-1}$ diverges

since $\sum \dfrac{3}{n}$ diverges by p-series test.

<u>Question</u> What do we do when we want to make comparisons but the inequalities are tedious or inconvenient?

<u>Limit comparison test</u> Suppose $a_n > 0$, $b_n > 0$ for all n.

If $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then

$\sum a_n$ and $\sum b_n$ either both converge or both diverge.

<u>Intuition</u> when $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c$

we're saying $a_n \approx c b_n$ for large n,

and so $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} c b_n$ should

have the same convergence behavior.

Example Use Limit Comparison Test to justify whether these series converge or diverge.

① $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2-5}{n^3+n+2}$

② $\displaystyle\sum_{n=1}^{\infty} \frac{1}{8n^4-3n^2}$

③ $\displaystyle\sum_{n=1}^{\infty} \frac{3n+2}{\sqrt{5n^3+3n-2}}$

① $$\lim_{n\to\infty} \frac{\frac{2n^2-5}{n^3+n+2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{2n^2-5}{n^3+n+2} \cdot \frac{n}{1}$$

$$= \lim_{n\to\infty} \frac{2n^3-5n}{n^3+n+2} = 2 > 0$$

So $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-5}{n^3+n+2}$ diverges by LCT since

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series test

② $$\lim_{n\to\infty} \frac{\frac{1}{8n^4-3n^2}}{\frac{1}{n^4}} = \lim_{n\to\infty} \frac{n^4}{8n^4-3n^2} = \frac{1}{8} > 0$$

So $\sum_{n=1}^{\infty} \frac{1}{n^4-7}$ converges by LCT

since $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges by p-series test

③ $$\lim_{n\to\infty} \frac{\frac{3n+2}{\sqrt{5n^3+3n-2}}}{\frac{1}{n^{1/2}}} = \lim_{n\to\infty} \frac{(3n+2)n^{1/2}}{\sqrt{5n^3+3n-2}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{(3n+2)^2}\cdot\sqrt{n}}{\sqrt{5n^3+3n-2}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{(3n+2)^2\cdot n}{5n^3+3n-2}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{(9n^2+12n+4)\cdot n}{5n^3+3n-2}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{9n^3+12n^2+4}{5n^3+3n-2}}$$

$$= \sqrt{\frac{9}{5}}$$