§7.4 Algebraic Identities

Our goal today is to integrate rational functions.

Example $\displaystyle\int \frac{1}{(x-2)(x-5)}\,dx$.

Method of Partial Fraction Decomposition

split the fraction into simpler pieces:

$$\frac{1}{(x-2)(x-5)} = \frac{A}{x-2} + \frac{B}{x-5}$$

Solve for A and B:

$$1 = \left(\frac{A}{x-2} + \frac{B}{x-5}\right)\left((x-2)(x-5)\right)$$

$$\Rightarrow \quad 1 = A(x-5) + B(x-2)$$

Plug in $x=5$: $\quad 1 = A(5-5) + B(5-2) \Rightarrow B = \frac{1}{3}$

Plug in $x=2$: $\quad 1 = A(2-5) + B(2-2) \Rightarrow A = -\frac{1}{3}$

Therefore, $\displaystyle\int \frac{1}{(x-2)(x-5)}\,dx = \int \left(\frac{-1/3}{x-2} + \frac{1/3}{x-5}\right)dx$

$$= -\frac{1}{3}\ln|x-2| + \frac{1}{3}\ln|x-5| + C.$$

<u>Example</u> $\displaystyle\int \frac{x+2}{x^2+x}\,dx$

factor denominator first!

$$\frac{x+2}{x^2+x} = \frac{x+2}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$$\Rightarrow \quad x+2 = A(x+1) + Bx$$

Plug in $x=-1$: $\quad -1+2 = A(-1+1) + B(-1) \Rightarrow B = -1$

Plug in $x=0$: $\quad 0+2 = A(0+1) + B(0) \Rightarrow A = 2$

$$\int \frac{x+2}{x^2+x}\,dx = \int \frac{2}{x}\,dx + \int \frac{-1}{x+1}\,dx$$

$$= 2\ln|x| - \ln|x+1| + C.$$

<u>Example</u> $\displaystyle\int \frac{10x-2x^2}{(x-1)^2(x+3)}\,dx$

$$\frac{10x-2x^2}{(x-1)^2(x+3)} = \underbrace{\frac{A}{x-1} + \frac{B}{(x-1)^2}} + \frac{C}{(x+3)}$$

$(x-1)^3$ would have three terms like this

$$\Rightarrow \quad 10x-2x^2 = A(x-1)(x+3) + B(x+3) + C(x-1)^2$$

Plug in $x=1$: $\quad 8 = 4B \Rightarrow B=2$

Plug in $x=-3$: $\quad -30-2(9) = 16C \Rightarrow C=-3$

Plug in $\underbrace{x=0}$: $\quad 0 = -3A + 3B + C = -3A + 6 - 3$

$\Rightarrow A = 1$

0 was arbitrary,
anything works

$$\int \frac{10x - 2x^2}{(x-1)^2(x+3)}\,dx = \int \frac{1}{x-1}\,dx + \underbrace{\int \frac{2}{(x-1)^2}\,dx}_{\substack{\text{careful!} \\ w = x-1}} + \int \frac{-3}{(x+3)}\,dx$$

$$= \ln|x-1| - 2(x-1)^{-1} - \ln|x+3| + C.$$

Example $\displaystyle\int \frac{2x^2 - x - 1}{(x^2+1)(x-2)}\,dx$

$$\frac{2x^2 - x - 1}{(x^2+1)(x-2)} = \underbrace{\frac{Ax+B}{x^2+1}}_{\text{unfactorable quadratics require this form}} + \frac{C}{x-2}$$

$$2x^2 - x - 1 = (Ax+B)(x-2) + C(x^2+1)$$

Plug in $x = 2$: $\quad 8 - 2 - 1 = 5C \Rightarrow C = 1$

Plug in $x = 0$: $\quad -1 = -2B + C = -1 = -2B + 1$

$\Rightarrow B = 1$

Plug in $\underbrace{x = 1}_{\text{arbitrary}}$: $\quad 2 - 1 - 1 = (A+B)(-1) + 2C$

$\Rightarrow 0 = -A - 1 + 2 \Rightarrow A = 1$

$$\int \frac{2x^2-x-1}{(x^2+1)(x-2)}\,dx = \int \frac{x+1}{x^2+1}\,dx + \int \frac{1}{x-2}\,dx$$

$w = x^2+1$

$dw = 2x\,dx$

$\frac{1}{2}dw = x\,dx$

$$= \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx + \int \frac{1}{x-2}\,dx$$

$$= \frac{1}{2}\int \frac{1}{w}\,dw + \arctan x + \ln|x-2| + C$$

$$= \frac{1}{2}\ln|x^2+1| + \arctan x + \ln|x-2| + C.$$

Rules for Partial Fraction Decomposition of $\frac{P(x)}{Q(x)}$

① We need $\deg P < \deg Q$ (otherwise need to do long division)

② if Q factors into distinct linear factors, use terms of the form $\frac{A}{x-c}$

③ if Q has a repeated linear factor $(x-c)^n$ use terms of the form $\frac{A_1}{x-c} + \frac{A_2}{(x-c)^2} + \cdots + \frac{A_n}{(x-c)^n}$

④ if Q has an unfactorable quadratic $g(x)$, use

$$\frac{Ax+B}{g(x)}$$

Problem. Find the partial fraction decomposition of each of the rational functions. Then find its indefinite integral.

a. $\frac{3x^2-8x+1}{(x-2)(x+1)(x-3)}$

b. $\frac{x+1}{6x+x^2}$

c. $\frac{1}{(x-1)^2(x-2)}$

d. $\frac{10x+2}{(x-5)(x^2+1)}$

e. $\frac{1}{(x-1)(x^2+1)}$

Remark. Remember to factor the denominator if it's not already done for you!

(a)

$$\frac{3x^2-8x+1}{(x-2)(x+1)(x-3)} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{x-3}$$

$$3x^2-8x+1 = A(x+1)(x-3) + B(x-2)(x-3) + C(x-2)(x+1)$$

$\underline{x=2}$ $\quad -3 = -3A \Rightarrow A = -1$

$\underline{x=-1}$ $\quad 12 = 12B \Rightarrow B = 1$

$\underline{x=3}$ $\quad 4 = 4C \Rightarrow C = 1$

$$= \int \frac{-1}{x-2}\,dx + \int \frac{1}{x+1}\,dx + \int \frac{1}{x-3}\,dx$$

$$= -\ln|x-2| + \ln|x+1| + \ln|x-3| + C$$

(b)

$$\frac{x+1}{x(x+6)} = \frac{A}{x} + \frac{B}{x+6}$$

$$x+1 = A(x+6) + Bx$$

$\underline{x=0}$ $\quad 1 = 6A \Rightarrow A = 1/6$

$\underline{x=-6}$ $\quad -5 = -6B \Rightarrow B = 5/6$

$$= \int \frac{1/6}{x}\,dx + \int \frac{5/6}{x+6}\,dx = \frac{1}{6}\ln|x| + \frac{5}{6}\ln|x+6| + C$$

(c) $$\frac{1}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$$

$$1 = A(x-1)(x-2) + B(x-2) + C(x-1)^2$$

$\underline{x=1}$ $\quad 1 = -B, \quad B = -1$

$\underline{x=2}$ $\quad 1 = C$

$\underline{x=3}$ $\quad 1 = 2A + B + 4C$

$\quad = 2A - 1 + 4, \quad A = -1$

$$\int \frac{-1}{x-1}\,dx + \int \frac{-1}{(x-1)^2}\,dx + \int \frac{1}{x-2}\,dx$$

$$= -\ln|x-1| + (x-1)^{-1} + \ln|x-2| + C$$

(d) $$\frac{10x+2}{(x-5)(x^2+1)} = \frac{A}{x-5} + \frac{Bx+C}{x^2+1}$$

$$10x+2 = A(x^2+1) + (Bx+C)(x-5)$$

$\underline{x=5}$ $\quad 52 = 26A \Rightarrow A = 2$

$\underline{x=0}$ $\quad 2 = A - 5C \Rightarrow C = 0$

$\underline{x=6}$ $\quad 62 = 37A + 6B$

$\quad 6B = 62 - 74$

$\quad B = -2$

$$= \int \frac{2}{x-5}\,dx + \int \frac{-2x}{x^2+1}\,dx$$

$$= 2\ln|x-5| + \int \frac{-dw}{w}$$

$w = x^2+1$

$dw = 2x\,dx$

$$= 2\ln|x-5| - \ln|x^2+1| + C.$$

(e) $$\frac{1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow \quad 1 = A(x^2+1) + (Bx+C)(x-1)$$

$\underline{x=1}$ $\quad 1 = 2A \Rightarrow A = 1/2$

$\underline{x=0}$ $\quad 1 = A - C \Rightarrow C = -1/2$

$\underline{x=2}$ $\quad 1 = 5A + 2B + C = \frac{5}{2} + 2B - \frac{1}{2}$

$$\Rightarrow 2B = -1 \Rightarrow B = -1/2$$

$$\int \frac{1}{(x-1)(x^2+1)}\,dx = \int \frac{1/2}{x-1}\,dx + \frac{1}{2}\int \frac{-x-1}{x^2+1}\,dx$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{2}\int \frac{x}{x^2+1}\,dx - \frac{1}{2}\int \frac{1}{x^2+1}\,dx$$

$w = x^2+1$

$dw = 2x\,dx$

$\frac{1}{2}dw = x\,dx$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{4}\int \frac{1}{w}\,dw - \arctan x + C$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{4}\ln|x^2+1| - \frac{1}{2}\arctan x + C.$$