8.4 Ratio Test

Question If $\sum a_n$ is a geometric series, what can be said about the ratio

$$\frac{a_{n+1}}{a_n}$$

of consecutive terms?

What condition on $\frac{a_{n+1}}{a_n}$ gives convergence?

With geometric series, the ratio of consecutive terms

$$\frac{a_{n+1}}{a_n} = \frac{ar^{n+1}}{ar^n} = r$$

is constant.

If $\left|\frac{a_{n+1}}{a_n}\right| = |r| < 1$, the series converges.

If $|r| \geq 1$, the series diverges.

Example Consider $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$.

Not geometric! But let's compute

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} \quad \text{where} \quad a_n = \frac{n}{e^n}$$

What do you think we can conclude?

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{n+1}{e^{n+1}}}{\frac{n}{e^n}}$$

$$= \lim_{n\to\infty} \frac{n+1}{e^{n+1}} \cdot \frac{e^n}{n}$$

$$= \lim_{n\to\infty} \frac{1}{e} \cdot \frac{n+1}{n} = \frac{1}{e} < 1$$

Conclusion In the long term, the terms of the series have approximately a common ratio less than 1, so we believe it behaves like a convergent geometric series.

Ratio test Consider $\sum_{n=1}^{\infty} a_n$ and let

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

- If $L < 1$, then $\sum a_n$ converges
- If $L > 1$, then $\sum a_n$ diverges
- If $L = 1$, the test is inconclusive.

Example What does the ratio test tell us about the following series:

① $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^3+1}$.

③ $\displaystyle\sum_{n=1}^{\infty} \frac{(-8)^n}{(2n)!}$

② $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$

④ $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+5}$

①

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^{n+1}}{(n+1)^3+1}\cdot\frac{n^3+1}{2^n}$$

$$= \lim_{n\to\infty}\frac{2^{n+1}}{2^n}\cdot\frac{n^3+1}{(n+1)^3+1}$$

$$= 2\lim_{n\to\infty}\frac{n^3+1}{(n+1)^3+1}$$

$$= 2 > 1$$

Therefore the series <u>diverges</u> by the ratio test

②

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(n+1)!}\ \frac{n!}{(-1)^n}\right|$$

$$= \lim_{n\to\infty}\frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty}\frac{n(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{(n+1)(n)(n-1)(n-2)\cdots 3\cdot 2\cdot 1}$$

$$= \lim_{n\to\infty}\frac{1}{n+1} = 0 < 1$$

Therefore the series <u>converges</u> by the ratio test

③ $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-8)^{n+1}}{(2(n+1))!}\cdot\frac{(2n)!}{(-8)^n}\right|$$

$$= 8\cdot\lim_{n\to\infty}\frac{(2n)!}{(2n+2)!}$$

$$= 8\cdot\lim_{n\to\infty}\frac{(2n)!}{(2n+2)(2n-1)(2n)!}$$

$$= 8\lim_{n\to\infty}\frac{1}{(2n+2)(2n-1)} = 0 < 1$$

Therefore the series converges by the ratio test

④ $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{(n+1)^3+5}\cdot\frac{n^3+5}{n^2}$$

$$= 1 \quad \text{(using ratio of leading coeff's)}$$

The ratio test is inconclusive. However, we can use the limit comparison test to prove the series diverges, comparing it with $\sum_{n=1}^{\infty}\frac{1}{n}$.