Problem 1. For each of the following series: (1) decide on a test to use, (2) use the test to determine whether the series converges or diverges, (3) write up a conclusion statement that makes clear the test used and your conclusion about convergence or divergence.

a. $\sum_{n=1}^{\infty} \frac{n!}{3^n}$

b. $\sum_{n=1}^{\infty} \frac{(-1/4)^n}{n}$

c. $\sum_{n=1}^{\infty} \frac{5}{7n^3-n^2+5}$ ← comparison test

d. $\sum_{n=1}^{\infty} \frac{2n^3+5n+3}{5n^3-n^2+5}$ ← n^{th} term test

(a) ratio test (due to factorial)

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!}$$

$$= \lim_{n\to\infty} \frac{n+1}{3} = \infty > 1$$

So the series diverges by ratio test.

(b) ratio test (due to exponential) but comparison test would work too.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1/4)^{n+1}}{n+1} \cdot \frac{n}{(-1/4)^n}\right|$$

$$= \lim_{n\to\infty} \frac{1}{4} \cdot \frac{n}{n+1} = \frac{1}{4} < 1$$

So the series converges by ratio test.

(d) $\lim_{n\to\infty} \frac{2n^3+5n+3}{5n^3-n^2+5} = \frac{2}{5} \neq 0$, so $\sum_{n=1}^{\infty} \frac{2n^3+5n+3}{5n^3-n^2+5}$

diverges by n^{th} term test.

Rules of Thumb so far

① nth term test is a good quick check for divergence but it often fails

② limit comparison test (or comparison test) is good for series that are "basically" p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ but have extra terms in numerator and denominator

③ ratio test is good for series with exponentials or factorials.

8.5 Alternating Series and Absolute Convergence

Example Let's consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \cdots$$

Do you think it converges or diverges?

Notice that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots$$

is just the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges

and it seems like

$$0 \le \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \le \sum_{n=1}^{\infty} \frac{1}{n^2}, \text{ so}$$

the alternating version should converge too.

Absolute convergence test

If $\sum |a_n|$ converges, then $\sum a_n$ converges

Warning if $\sum |a_n|$ diverges, test is inconclusive.

Example What does the absolute convergence test tell us about:

① $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ Converges since $\sum \frac{1}{n^3}$ converges

② $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{5/2}}$ converges since $\sum \frac{1}{n^{5/2}}$ converges

③ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ inconclusive since $\sum \frac{1}{n}$ diverges

④ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ inconclusive since $\sum \frac{1}{\sqrt{n}}$ diverges.

Alternating series test

Consider $\sum_{n=1}^{\infty} a_n$

where $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$,

and $b_n \geq 0$ for all n.

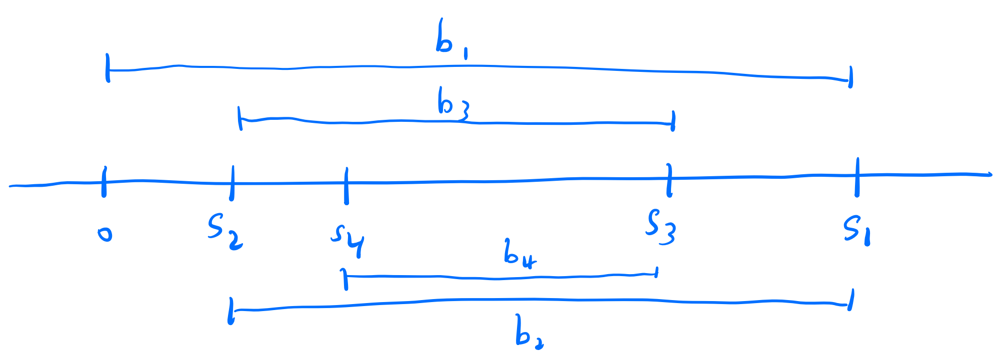
Suppose ① $b_{n+1} \leq b_n$ for all n

(meaning b_n is always decreasing)

② $\lim_{n \to \infty} b_n = 0$

Then $\sum_{n=1}^{\infty} a_n$ converges.

"Proof" $S_1 = b_1$, $S_2 = b_1 - b_2$, $S_3 = b_1 - b_2 + b_3$, $S_4 = b_1 - b_2 + b_3 - b_4, \ldots$



We've zeroing in on a limit.

Examples Explain why the following series converges using the alternating series test.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n+5}$$

Let $b_n = \frac{1}{3n+5}$. Then

① b_n is decreasing since

$$b_{n+1} = \frac{1}{3(n+1)+5} \leq \frac{1}{3n+5} = b_n$$

② $\lim_{n\to\infty} \frac{1}{3n+5} = 0$

So the series $\sum (-1)^{n+1} \frac{1}{3n+5}$ converges by alternating series test.