8.5 Absolute Convergence and Alternating Series Tests, continued

Last time, we learned that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges by the alternating series test, but it's conditionally convergent since $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Fact Later, we'll learn that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

converges to $\ln(2) \approx 0.69\dots$

A curious demonstration Let's rearrange so that each positive term is followed by two negatives:

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \dots$$

Then let's add parentheses for emphasis:

$$\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\cdots$$

And notice how this simplifies:

$$\left(\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{10}\right)-\frac{1}{12}+\cdots$$

This is our original series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$

except halved! So it now converges to $\frac{1}{2}\ln 2$!

<u>Theorem</u> (Riemann Rearrangement Theorem)

Consider a conditionally convergent series.

Then for any real number L, the order of the terms of the series can be rearranged so it converges to L.

<u>Remark</u> This cannot be done with absolutely convergent series.