Math 102 — Partial fractions

Summary. Try each of the following problems together in a small group.

Problem 1. For each of the following rational functions, split the function into partial fractions.

a. $\frac{x+1}{6x+x^2}$ *Hint: split into 2 fractions*

b. $\frac{3x^2-8x+1}{(x-2)(x+1)(x-3)}$ *Hint: split into 3 fractions* $\frac{A}{x-2}+\frac{B}{x+1}+\frac{C}{x-3}$

c. $\frac{1}{x^3-x^2}$ *Hint: split into 3 fractions*

d. $\frac{10x+2}{x^3-5x^2+x-5}$ *Hint: the denominator will factor as* $(x-5)(x^2+1)$*; note the quadratic cannot be factored, so the rule we'll use is to split this into* $\frac{A}{x-5}+\frac{Bx+C}{x^2+1}$

Problem 2. Find the indefinite integral of each of the first three functions above. Try the fourth too, but you might get stuck so we'll be sure to talk about that one together.

a) $\dfrac{x+1}{x(x+6)} = \dfrac{A}{x} + \dfrac{B}{x+6}$

$x+1 = A(x+6) + Bx$

$x=0 \qquad 1 = 6A \Rightarrow A = 1/6$

$x=-6 \qquad -5 = -6B \Rightarrow B = 5/6$

$= \int \frac{1/6}{x}\,dx + \int \frac{5/6}{x+6}\,dx = \frac{1}{6}\ln|x| + \frac{5}{6}\ln|x+6| + C$

b) $$\frac{3x^2-8x+1}{(x-2)(x+1)(x-3)} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{x-3}$$

$$3x^2-8x+1 = A(x+1)(x-3) + B(x-2)(x-3) + C(x-2)(x+1)$$

$x=2$ $\quad -3 = -3A \Rightarrow A = -1$

$x=-1$ $\quad 12 = 12B \Rightarrow B = 1$

$x=3$ $\quad 4 = 4C \Rightarrow C = 1$

$$= \int \frac{-1}{x-2}\,dx + \int \frac{1}{x+1}\,dx + \int \frac{1}{x-3}\,dx$$

$$= -\ln|x-2| + \ln|x+1| + \ln|x-3| + C$$

c) $$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$$1 = Ax(x-1) + B(x-1) + Cx^2$$

$x=0$ $\quad 1 = -B \Rightarrow B = -1$

$x=1$ $\quad 1 = C$

$x=2$ $\quad 1 = 2A + B + 4C \Rightarrow 1 = 2A - 1 + 4 \Rightarrow A = -1$

$$= \int \frac{-1}{x}\,dx + \int \frac{-1}{x^2}\,dx + \int \frac{1}{x-1}\,dx$$

$$= -\ln|x| + x^{-1} + \ln|x-1| + C$$

d) $$\frac{10x+2}{(x-5)(x^2+1)} = \frac{A}{x-5} + \frac{Bx+C}{x^2+1}$$

$$10x+2 = A(x^2+1) + (Bx+C)(x-5)$$

$x=5$ $\quad 52 = 26A \Rightarrow A = 2$

$x=0$ $\quad 2 = A - 5C \Rightarrow C = 0$

$x=6$ $\quad 62 = 37A + 6B$

$$6B = 62 - 74$$

$$B = -2$$

$$= \int \frac{2}{x-5}\,dx + \int \frac{-2x}{x^2+1}\,dx$$

$$= 2\ln|x-5| + \int \frac{-du}{u} \qquad u = x^2+1,\ du = 2x\,dx$$

$$= 2\ln|x-5| - \ln|x^2+1| + C.$$