Math 102 — Improper integrals

Summary. Try each of the following problems together in a small group.

Problem 1. Determine whether the following improper integrals converge.

a. $\int_0^\infty e^{-2x}\,dx$

b. $\int_{-1}^1 \frac{1}{x^4}\,dx$

c. $\int_1^\infty \frac{1}{(2x+5)^2}\,dx$

d. $\int_1^\infty \frac{1}{(2x-5)^2}\,dx$

(a)

$$\int_0^\infty e^{-2x}\,dx = \lim_{b\to\infty} \int_0^b e^{-2x}\,dx$$

$$= \lim_{b\to\infty} -\frac{1}{2}e^{-2x}\Big|_0^b$$

$$= \lim_{b\to\infty} \frac{1}{2}\left(1 - e^{-2b}\right)$$

$$= \lim_{b\to\infty} \frac{1}{2}\left(1 - \frac{1}{e^{2b}}\right)$$

$$= \frac{1}{2}$$

(b) $\int_{-1}^{1} \frac{1}{x^4}\,dx = \int_{0}^{1} \frac{1}{x^4}\,dx + \int_{-1}^{0} \frac{1}{x^4}\,dx$

$$\int_0^1 \frac{1}{x^4}\,dx = \lim_{a\to 0^+} \int_a^1 x^{-4}\,dx$$

$$= \lim_{a\to 0^+} -\frac{1}{3}x^{-3}\Big|_a^1$$

$$= \lim_{a\to 0^+} \frac{1}{3}\left(a^{-3} - 1\right)$$

$$= \lim_{a\to 0^+} \frac{1}{3}\left(\frac{1}{a^3} - 1\right)$$

$= \infty$ so the integral diverges

(c) $\int_1^{\infty} \frac{1}{(2x+5)^2}\,dx$ $\quad u = 2x+5$

$du = 2dx$

$= \frac{1}{2}\int_7^{\infty} \frac{1}{u^2}\,du$ $\quad \frac{1}{2}du = dx$

$$= \frac{1}{2}\lim_{b\to\infty} \int_7^b u^{-2}\,du$$

$$= \frac{1}{2}\lim_{b\to\infty} -u^{-1}\Big|_7^b$$

$$= \frac{1}{2}\lim_{b\to\infty} \frac{1}{7} - \frac{1}{b} = \frac{1}{14}$$

ⓓ $\int_1^\infty \frac{1}{(2x-5)^2}$

$u = 2x-5$

$du = 2dx$

$\frac{1}{2}du = dx$

$= \frac{1}{2}\int_{-3}^{\infty} \frac{1}{u^2}\,du$

$= \frac{1}{2} \int_{-3}^{0} \frac{1}{u^2}\,du + \int_0^1 \frac{1}{u^2}\,du + \int_1^\infty \frac{1}{u^2}\,du$

$\int_{-3}^{0} \frac{1}{u^2}\,du = \lim_{b\to 0^-} \int_{-3}^{b} u^{-2}\,du$

$= \lim_{b\to 0^-} -u^{-1} \Big|_{-3}^{b}$

$= \lim_{b\to 0^-} -\left(b^{-1} - (-3)^{-1}\right)$

$= \lim_{b\to 0^-} -\frac{1}{b} - \frac{1}{3}$

$= \infty$, so the whole integral diverges.