Math 102 — Taylor polynomials

Summary. Try each of the following problems together in a small group.

Problem 1. Find the degree n Taylor polynomial of $f(x)$ using the formula

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

a. $f(x) = \ln(1-x), n = 4$

b. $f(x) = \frac{1}{1-x}, n = 4$

c. $f(x) = (x+1)^{1/3}, n = 4$

Problem 2. Use substitution into a known Taylor approximation to find a Taylor approximation with 4 terms for the given function.

a. $f(x) = \sin x^2$

b. $f(x) = \frac{1}{1-x^3}$

c. $f(x) = \ln(1 + \cancel{\sqrt{x}})$

x

Problem 1

a) $f(x) = \ln(1-x)$ $\qquad f(0) = 0$

$f'(x) = \dfrac{-1}{1-x} = -(1-x)^{-1}$ $\qquad f'(0) = -1$

$f''(x) = -(1-x)^{-2}$ $\qquad f''(0) = -1$

$f'''(x) = -2(1-x)^{-3}$ $\qquad f'''(0) = -2$

$f^{(4)}(x) = -6(1-x)^{-4}$ $\qquad f^{(4)}(0) = -6$

$$P_4(x) = -x - \frac{1}{2!}x^2 - \frac{2}{3!}x^3 - \frac{6}{4!}x^4$$

$$= -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4$$

b) $f(x) = \dfrac{1}{1-x} = (1-x)^{-1}$ $\qquad f(0) = 1$

$f'(x) = (1-x)^{-2}$ $\qquad f'(0) = 1$

$f''(x) = 2(1-x)^{-3}$ $\qquad f''(0) = 2$

$f'''(x) = 6(1-x)^{-4}$ $\qquad f'''(0) = 6$

$f^{(4)}(x) = 24(1-x)^{-5}$ $\qquad f^{(4)}(0) = 24$

$$P_4(x) = 1 + x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 + \frac{24}{4!}x^4$$

$$= 1 + x + x^2 + x^3 + x^4$$

c) $f(x) = (x+1)^{1/3}$ $\quad\quad f(0) = 1$

$f'(x) = \frac{1}{3}(x+1)^{-2/3}$ $\quad\quad f'(0) = \frac{1}{3}$

$f''(x) = \left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)(x+1)^{-5/3}$ $\quad\quad f''(0) = \frac{-2}{9}$

$= \frac{-2}{9}(x+1)^{-5/3}$

$f'''(x) = \frac{10}{27}(x+1)^{-8/3}$ $\quad\quad f'''(0) = \frac{10}{27}$

$f^{(4)}(x) = \frac{-80}{81}(x+1)^{-11/3}$ $\quad\quad f^{(4)}(0) = \frac{-80}{81}$

$$P_4(x) = 1 + \frac{1}{3}x - \frac{2/9}{2!}x^2 + \frac{10/27}{3!}x^3 - \frac{80/81}{4!}x^4$$

(could simplify this, but nothing too insightful pops out)

Problem 2

a) $f(x) = \sin x^2$

Plug x^2 into

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

and get

$$\sin x^2 \approx x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!}$$

$$= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!}$$

b) Plug x^3 into

$$\frac{1}{1-x} \approx 1 + x + x^2 + x^3$$

and get

$$\frac{1}{1-x^3} \approx 1 + x^3 + (x^3)^2 + (x^3)^3$$

$$= 1 + x^3 + x^6 + x^9$$

c) Plug $-x$ into

$$\ln(1-x) \approx -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4$$

and get

$$\ln(1+x) \approx -(-x) - \frac{1}{2}(-x)^2 - \frac{1}{3}(-x)^3 - \frac{1}{4}(-x)^4$$

$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$$