Math 102 — Geometric series

Summary. Try each of the following problems together in a small group.

Problem 1. For each infinite geometric series, find the value of its sum if the series converges.

a. $(1/3)^4 + (1/3)^5 + (1/3)^6 + \cdots$

b. $5 - \frac{5}{4} + \frac{5}{4^2} - \frac{5}{4^3} + \cdots$

c. $20 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \cdots$

d. $\sum_{n=0}^{\infty} \frac{2+4^n}{7^n}$

Problem 2. At the start of every year, Phuong invests \$5000 into her savings account. The investment yields 2% interest, compounded yearly. This means at the end of the first year, her account will be worth $Q_1 = 5000(1.02)$ dollars.

a. How much will her account be worth at the end of year 2?

b. How much will her account be worth at the end of year 3?

c. At the end of year 30?

Problem 3. The **Sierpinski carpet** is a cool two-dimensional fractal set. It is constructed by removing the center one-ninth of a square of side-length 1, then removing the centers of the eight smaller remaining squares, and so on *forever*. See the figure below for the first 3 steps in the construction.

a. How much area has been removed after the first step of the construction?

b. How much area has been removed after the second step?

c. After the third step?

d. How much area gets in total if the process never stops?

Figure 1: The first three steps of the construction of the Sierpinski carpet.

Problem 1. For each infinite geometric series, find the value of its sum if the series converges.

a. $(1/3)^4 + (1/3)^5 + (1/3)^6 + \cdots$

b. $5 - \frac{5}{4} + \frac{5}{4^2} - \frac{5}{4^3} + \cdots$

c. $20 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \cdots$

d. $\sum_{n=0}^{\infty} \frac{2+4^n}{7^n}$

a) $\left(\frac{1}{3}\right)^4 \left[1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots\right]$

$= \left(\frac{1}{3}\right)^4 \cdot \left[\frac{1}{1-\frac{1}{3}}\right] = \frac{1}{54}$

b) $5\left[1 - \frac{1}{4} + \frac{1}{4^2} - \frac{1}{4^3} + \cdots\right]$

$= 5\left[1 + \left(-\frac{1}{4}\right) + \left(-\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right)^3 + \cdots\right]$

$= 5\left[\frac{1}{1-\left(-\frac{1}{4}\right)}\right]$

$= 4$

c) $20 + \frac{1}{3}\left[1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \cdots\right]$

$= 20 + \frac{1}{3} \cdot \frac{1}{1 - \frac{1}{3}}$

$= 20.5$

d) $\sum_{n=0}^{\infty} \frac{2}{7^n} + \sum_{n=0}^{\infty} \left(\frac{4}{7}\right)^n$

$= 2\left[1 + \frac{1}{7} + \left(\frac{1}{7}\right)^2 + \cdots\right] + \left[1 + \frac{4}{7} + \left(\frac{4}{7}\right)^2 + \cdots\right]$

$= 2\left[\frac{1}{1-\frac{1}{7}}\right] + \frac{1}{1 - \frac{4}{7}}$

$= \frac{14}{3}$

Problem 2. At the start of every year, Phuong invests \$5000 into her savings account. The investment yields 2% interest, compounded yearly. This means at the end of the first year, her account will be worth $Q_1 = 5000(1.02)$ dollars.

a. How much will her account be worth at the end of year 2?

b. How much will her account be worth at the end of year 3?

c. At the end of year 30?

a) $$Q_2 = 5000(1.02) + Q_1(1.02)$$

$$= \underbrace{5000(1.02)}_{\text{new investment}} + \underbrace{5000(1.02)^2}_{\text{interest from old investment}}$$

b)

$$Q_3 = \underbrace{5000(1.02)}_{\text{new}} + \underbrace{Q_2(1.02)}_{\text{old}}$$

$$= 5000(1.02) + 5000(1.02)^2 + 5000(1.02)^3$$

c) $$Q_{30} = 5000(1.02) + 5000(1.02)^2 + \dots + 5000(1.02)^{30}$$

$$= 5000(1.02)\left[1 + (1.02) + \dots + (1.02)^{29}\right]$$

$$= 5000(1.02)\left[\frac{1-(1.02)^{29+1}}{1-(1.02)}\right]$$

Problem 3. The **Sierpinski carpet** is a cool two-dimensional fractal set. It is constructed by removing the center one-ninth of a square of side-length 1, then removing the centers of the eight smaller remaining squares, and so on *forever*. See the figure below for the first 3 steps in the construction.

a. How much area has been removed after the first step of the construction?

b. How much area has been removed after the second step?

c. After the third step?

d. How much area gets in total if the process never stops?

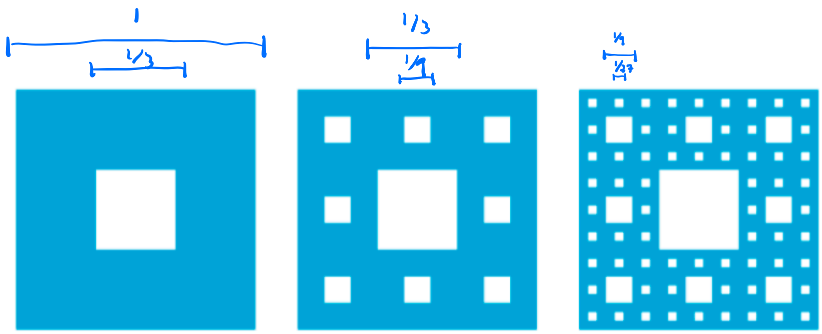

Figure 1: The first three steps of the construction of the Sierpinski carpet.

a) $A_1 = \left(\frac{1}{3}\right)^2$

b) $A_2 = \frac{1}{9} + 8\left(\frac{1}{9}\right)^2$

c) $A_3 = \frac{1}{9} + 8\left(\frac{1}{9}\right)^2 + 64\left(\frac{1}{27}\right)^2$

$= \frac{1}{9} + 8\left(\frac{1}{9}\right)^2 + 8^2\left(\frac{1}{9}\right)^3$

d) $\frac{1}{9} + 8\left(\frac{1}{9}\right)^2 + 8^2\left(\frac{1}{9}\right)^3 + \cdots$

$= \frac{1}{9}\left[1 + \frac{8}{9} + \left(\frac{8}{9}\right)^2 + \cdots\right] = \frac{1}{9} \cdot \frac{1}{1 - \frac{8}{9}} = 1.$