12.6 Directional Derivatives

Goal We've learned how partial derivatives f_x and f_y tells us about how f changes as we move in x or y directions. We'd like to define and study derivatives (i.e. rates of change) of $f(x,y)$ in any arbitrary direction given by a unit vector $\vec{u} \in \mathbb{R}^2$.

Def Let $\vec{u} = \langle u_1, u_2 \rangle \in \mathbb{R}^2$ be a unit vector. The directional derivative of f in the direction of $\vec{u}$ is given by $D_{\vec{u}} f = u_1 f_x + u_2 f_y$

(it's a combination of f_x and f_y)

Example Let $f(x,y) = x^2 + 2y^2$ and compute $D_u f(1,1)$ when $\vec{u}$ points in the direction of

① $\vec{v} = \langle 1, 1 \rangle$ ② $\vec{v} = \langle -1, 1 \rangle$ ③ $\vec{v} = \langle -2, -2 \rangle$

① $\vec{u} = \frac{1}{\|\vec{v}\|} \vec{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

$f_x = 2x, \quad f_y = 4y$

$f_x(1,1) = 2, \quad f_y(1,1) = 4$

$D_{\vec{u}} f(1,1) = \frac{1}{\sqrt{2}}(2) + \frac{1}{\sqrt{2}}(4)$

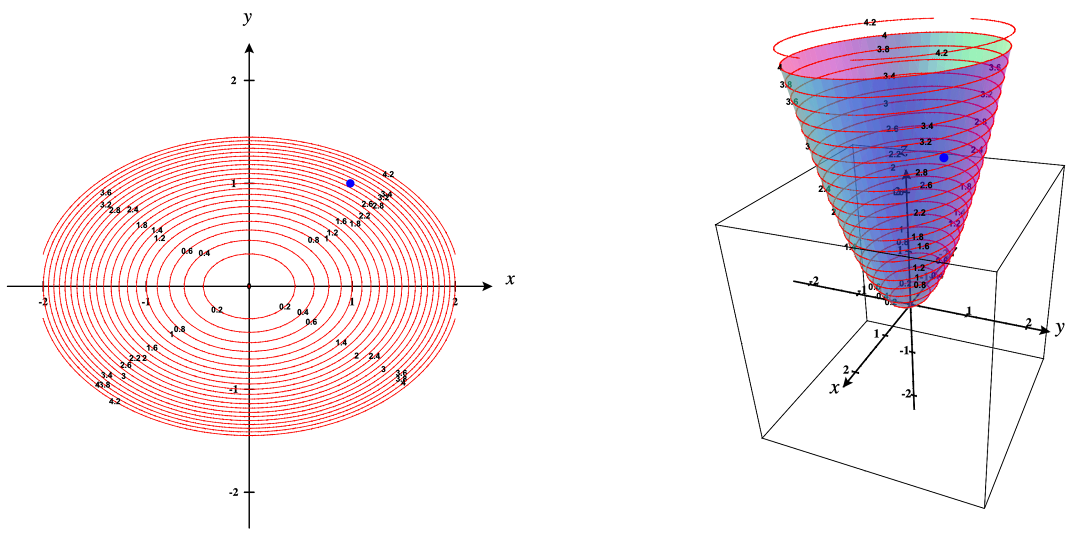
$= \frac{6}{\sqrt{2}}$

(2) $\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \left\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

$$D_{\vec{u}} f(1,1) = \frac{-1}{\sqrt{2}}(2) + \frac{1}{\sqrt{2}}(4) = \frac{2}{\sqrt{2}}$$

(3) $\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \left\langle \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\rangle$

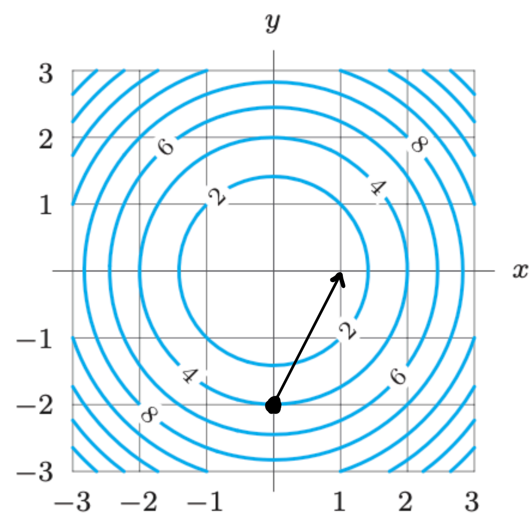
$$D_{\vec{u}} f(1,1) = \frac{-1}{\sqrt{2}}(2) - \frac{1}{\sqrt{2}}(4) = \frac{-6}{\sqrt{2}}$$



Contour plot and graph of $f(x,y) = x^2 + 2y^2$

Example Given the contour plot of f below, determine the sign of $D_{\vec{u}}f(0,-2)$ when $\vec{u}$ is in the direction of

1. $\langle 1, 2 \rangle$
2. $\langle 1, -2 \rangle$
3. $\langle -1, -1 \rangle$
4. $\langle -1, 0 \rangle$
5. $\langle 1, 0 \rangle$



(Important) **Question** Given a contour plot which direction should $\vec{u}$ point so that $D_{\vec{u}}f(a,b) = 0$?

tangent to the level curve passing through (a,b)

Def The <u>gradient</u> <u>vector</u> of f at (a,b) is

the vector $\nabla f(a,b) = \langle f_x(a,b), f_x(a,b) \rangle$.

Facts

① $D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}$

② $D_{\vec{u}} f(a,b) = \| \nabla f(a,b) \| \cos\theta$ where

θ is the angle between $\vec{u}$ and $\nabla f(a,b)$

③ Among all choices of $\vec{u}$, $D_{\vec{u}} f(a,b)$

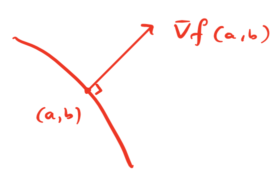
is

ⓐ <u>0</u> when $\vec{u}$ is perpendicular to gradient vector $\theta = \pi/2$

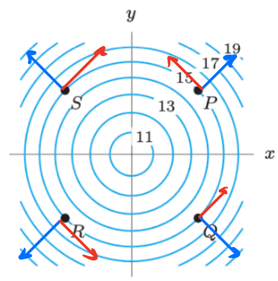
ⓑ <u>maximized</u> when $\vec{u}$ points in the direction of $\nabla f(a,b)$ $\theta = 0$

ⓒ <u>minimized</u> when $\vec{u}$ points in direction opposite $\nabla f(a,b)$ $\theta = \pi$

④ $\nabla f(a,b)$ is perpendicular to the level curve of f at (a,b)



Problem 1. In the contour plot below sketch the direction of ∇f at each of the points P, Q, R, and S. Also sketch a direction $\mathbf{u}$ where $D_{\mathbf{u}} f$ is zero at each of these points.



red vectors are tangent to level curve $\Rightarrow D_u f = 0$

blue vectors are perpendicular and facing direction of increase.

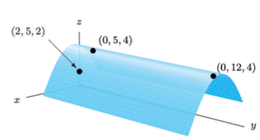
Problem 2. Consider the graph of a function $f(x, y)$ shown below. Give sign of the following directional derivatives

a. $D_{\mathbf{u}} f(2,5)$ where $\mathbf{u} = \langle -1, 0\rangle$ $+$

b. $D_{\mathbf{u}} f(2,5)$ where $\mathbf{u} = \langle 1/\sqrt{2}, 1/\sqrt{2}\rangle$ $-$

c. $D_{\mathbf{u}} f(0,5)$ where $\mathbf{u} = \langle 0, 1\rangle$ 0

d. $D_{\mathbf{u}} f(0,12)$ where $\mathbf{u} = \langle 1/\sqrt{2}, -1/\sqrt{2}\rangle$ $-$



Problem 3. Let $f(x,y) = -x^2y + xy^2 + xy$ and $P = (2,1)$. Compute $D_{\mathbf{u}}f(P)$ for each unit vector $\mathbf{u}$ given below.

a. $\mathbf{u}$ in the direction of $\mathbf{v} = \langle 3,4 \rangle$

b. $\mathbf{u}$ in the direction from P to $Q = (1,-1)$

c. $\mathbf{u}$ in the direction of maximum rate of change

d. $\mathbf{u}$ in the direction of minimum (ie. most negative) rate of change

e. $\mathbf{u}$ in the direction perpendicular to $\nabla f(P)$

$\nabla f = \langle -2xy + y^2 + y, -x^2 + 2xy + x \rangle$

$\nabla f(P) = \langle -2, 2 \rangle$

(a) $\vec{u} = \langle 3/5, 4/5 \rangle \quad D_{\vec{u}} f(P) = \nabla f(P) \cdot \vec{u} = \frac{-6}{5} + \frac{8}{5} = \frac{2}{5}$

(b) $\vec{PQ} = \langle -1, -2 \rangle, \ \vec{u} = \frac{1}{\sqrt{5}} \langle -1, -2 \rangle, \quad D_{\vec{u}} f(P) = \frac{2}{\sqrt{5}} - \frac{4}{\sqrt{5}} = \frac{-2}{\sqrt{5}}$

(c) $D_{\vec{u}} f(P) = \| \nabla f(P) \| = \sqrt{8}$

(d) $D_{u} f(P) = -\| \nabla f(P) \| = -\sqrt{8}$

(e) $D_{u} f(P) = 0$

Problem 4. Repeat the previous problem using $f(x,y) = x^2 + 2y^2 - xy - 7x$ and $P = (1,1)$.

$\nabla f = \langle 2x - y - 7, 4y - x \rangle$

$\nabla f(P) = \langle -6, 3 \rangle$

(a) $D_{\vec{u}} f(P) = \nabla f(P) \cdot \langle \frac{3}{5}, \frac{4}{5} \rangle = \frac{-18}{5} + \frac{12}{5} = \frac{-6}{5}$

$\vec{PQ} = \langle 0, -2 \rangle$

$\vec{u} = \langle 0, -1 \rangle$

(b) $D_{\vec{u}} f(P) = \nabla f(P) \cdot \langle 0, -1 \rangle = -3$

(c) $D_{\vec{u}} f(P) = \| \nabla f(P) \| = \sqrt{45}$

(d) $D_{\vec{u}} f(P) = -\| \nabla f(P) \| = -\sqrt{45}$

(e) $D_{\vec{u}} f(P) = 0$