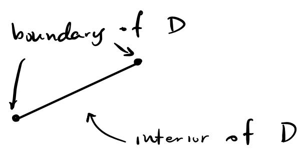
12.8 Extreme Values (Global optimization)

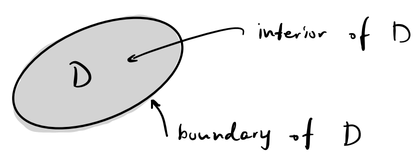
Calc I reminder (Extreme Value Theorem) If $f(x)$ is continuous on $[a,b]$ it has a global max and min which occur at critical points in (a,b) or at endpoints of $[a,b]$.

Calc III version (Extreme Value Theorem) If $f(x,y)$ is continuous on a bounded region $D \subseteq \mathbb{R}^2$ it has a global max and min which occur at critical points in the interior of D or on the boundary of D.

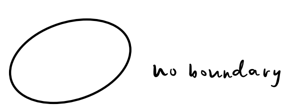
Examples of D



D = line segment

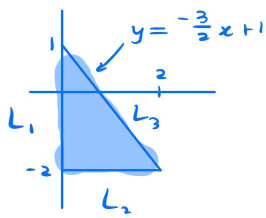


D = filled in 2d region



D = closed curve / "loop"

Example Let $f(x,y) = x^2 - y^2 + 5$ and let D be the region enclosed by the triangle with vertices $(0,-2)$, $(0,1)$, and $(2,-2)$. Find the global min and max of f over this region.



Step 1 find critical points in the interior of D:

$\nabla f = \langle 2x, -2y \rangle$, so $(0,0)$ is the only critical point and $\boxed{f(0,0) = 5}$

Step 2 find min, max of f along each boundary segment

L_1 Notice $x = 0$ and $-2 \le y \le 1$, so we want to optimize

$$g_1(y) = f(0,y)$$
$$= -y^2 + 5 \text{ over } [-2,1]$$

$g_1'(y) = -2y$, so $y = 0$ is critical point of g_1

and

testing endpoints

$$\boxed{\begin{array}{l} g_1(-2) = 1 \\ g_1(0) = 5 \\ g_1(1) = 4 \end{array}}$$

min on L_1

max on L_1

(L_2) Notice $y=-2$, $0 \le x \le 2$, so we want to optimize $g_2(x) = f(x,-2) = x^2 - 4 + 5$

$= x^2 + 1$ over $[0,2]$

$g_2'(x) = 2x$, so $x=0$ is only critical point of g_2

and

$$\boxed{\begin{aligned} g_2(0) &= 1 \quad \text{min on } L_2 \\ g_2(2) &= 5 \quad \text{max on } L_2 \end{aligned}}$$

(L_3) Here $y = -\frac{3}{2}x + 1$ and $0 \le x \le 2$, so we want to optimize

$$g_3(x) = f(x, -\tfrac{3}{2}x+1)$$

$$= x^2 - \left(-\tfrac{3}{2}x+1\right)^2 + 5$$

$$= -\tfrac{5}{4}x^2 + 3x + 4 \quad \text{over } [0,2]$$

$g_3'(x) = -\frac{5}{2}x + 3$, so $x = \frac{6}{5}$ is critical point of g_3 and

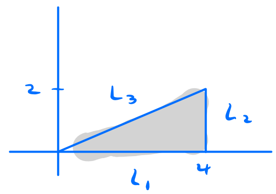
$$\boxed{\begin{aligned} g_3\left(\tfrac{6}{5}\right) &= 5.8 \quad \text{min on } L_3 \\ g_3(0) &= 4 \quad \text{max on } L_3 \\ g_3(2) &= 5 \end{aligned}}$$

Step 3 Compare all values found to get global min, max.

Global min value is 1, which occurs at $(0,-2)$

Global max value is 5.8, which occurs at $\left(\frac{6}{5}, \frac{-4}{5}\right)$

Problem 1. Let $f(x, y) = x^2 - 2xy + 4y^2 - 4x + 24$. Find the global extrema of f on the domain whose boundary is given by the triangle with vertices $(0, 0), (4, 0)$, and $(4, 2)$.



$\nabla f = \langle 2x - 2y - 4, 8y - 2x \rangle$, $\nabla f = \vec{0}$ when

$$\begin{cases} 2x - 2y - 4 = 0 \\ 8y - 2x = 0 \end{cases} \Rightarrow \begin{cases} x = y + 2 \\ x = 4y \end{cases} \Rightarrow 4y = y + 2$$

$\Rightarrow y = \frac{2}{3}, x = \frac{8}{3}$

$f(\frac{8}{3}, \frac{2}{3}) = \frac{56}{3} \approx 18.67$

L_1 $y = 0$, $0 \leq x \leq 4$, so we want to optimize

$g_1(x) = f(x, 0) = x^2 - 4x + 24$ over $[0, 4]$

$g_1'(x) = 2x - 4 \Rightarrow x = 2$ is critical point

$$\boxed{\begin{aligned} g_1(0) &= 24 \\ g_1(2) &= 20 \\ g_1(4) &= 24 \end{aligned}}$$

L_2 $x = 4$, $0 \leq y \leq 2$, so we want to optimize

$g_2(y) = f(4, y) = 16 - 8y + 4y^2 - 16 + 24$

$= 4y^2 - 8y + 24$ over $[0, 2]$

$g_2'(y) = 8y - 8 \Rightarrow y = 1$ is critical point

$$\boxed{\begin{aligned} g_2(0) &= 24 \\ g_2(1) &= 20 \\ g_2(2) &= 24 \end{aligned}}$$

L_3 $y = \frac{1}{2}x$, $0 \leq x \leq 4$, so we want to optimize

$g_3(x) = f(x, \frac{1}{2}x) = x^2 - 2x(\frac{1}{2}x) + 4(\frac{1}{2}x)^2 - 4x + 24$

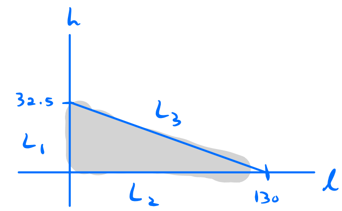
$= x^2 - 4x + 24$ over $[0, 4]$

we did this already with L_1.

So global min value is 18.67, max value is 24.

Problem 2. The sum of the *length* and *girth* of a rectangular box cannot exceed 130 inches. The *girth* of a box is defined to be twice the sum of its *width* and *height*. Assuming you want to make a box with equal width and height, find the maximum possible volume of such a box under the given constraints.

$l + g \le 130, \quad g = 2(w+h), \quad w = h$



$\Rightarrow \quad l + g \le 130, \quad g = 4h,$

$\Rightarrow \quad l + 4h \le 130, \quad h, l > 0$

$\text{volume} = f(l, h) = h^2 l$

$$\nabla f = \langle h^2, 2hl \rangle, \quad \nabla f = \vec{0} \text{ when } \begin{cases} h^2 = 0 \\ 2hl = 0 \end{cases}$$

$\Rightarrow$ points along L_2 are all critical points (where $h = 0$), but this is not a valid dimension for a box (ie. not in the domain of f) so we ignore these.

So max must occur on L_3

Here, $l = 130 - 4h$, $0 < h < 32.5$ so we want to optimize

$$g(h) = f(130 - 4h, h) \text{ over } (0, 32.5)$$
$$= h^2(130 - 4h)$$
$$= 130h^2 - 4h^3$$

$$g'(h) = 260h - 12h^2,$$

$\Rightarrow \quad h(260 - 12h) = 0$ gives critical points

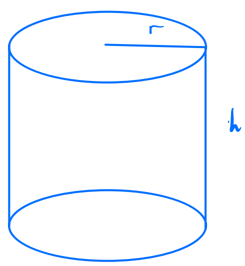
$\Rightarrow \quad h = 0$ (not valid), $h = \frac{260}{12} = \frac{65}{3}$,

$l = 130 - 4\left(\frac{65}{3}\right) = \frac{130}{3}$

So max volume is

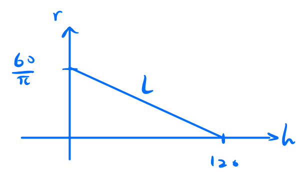
$$f\left(\frac{130}{3}, \frac{65}{3}\right) = \boxed{\left(\frac{65}{3}\right)^2\left(\frac{130}{3}\right)}$$

Problem 3. Find the maximum volume of a cylindrical soda can such that the sum of its height and circumference is 120 centimeters.



$h + 2\pi r = 120, \quad h, r > 0$

$\text{volume} = f(h, r) = \pi r^2 h$



want to maximize f over the line L given by $h + 2\pi r = 120$ where $h, r > 0$.

$$\nabla f = \langle \pi r^2, 2\pi r h \rangle = \vec{0} \implies \begin{cases} \pi r^2 = 0 \\ 2\pi r h = 0 \end{cases} \implies \begin{cases} r = 0 \\ r = 0 \text{ or } h = 0 \end{cases}$$

$\implies$ no critical points on L

We want to maximize

$$g(r) = f(120 - 2\pi r, r) \quad \text{over} \quad \left(0, \frac{60}{\pi}\right)$$

$$= \pi r^2 (120 - 2\pi r)$$

$$= 120\pi r^2 - 2\pi^2 r^3$$

$$g'(r) = 240\pi r - 6\pi^2 r^2$$

$$= 6\pi r(40 - \pi r)$$

$= 0$ when

$r = 0$ (not valid), $\quad r = \frac{40}{\pi}, \quad h = 120 - 2\pi\left(\frac{40}{\pi}\right) = 40$

So max volume is

$$f\left(40, \frac{40}{\pi}\right) = \pi\left(\frac{40}{\pi}\right)^2 40$$

$$= \boxed{\frac{40^3}{\pi}}$$