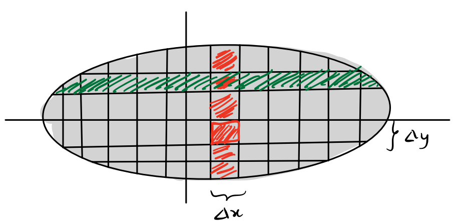
13.1 Introduction to Double Integrals

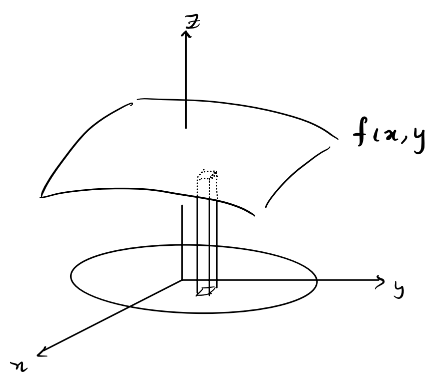
Def Given a region $R \subseteq \mathbb{R}^2$ in the xy-plane and a function $f(x,y)$, the double integral of f over R is

$$\iint_R f(x,y)\,dA = \lim_{\Delta x, \Delta y \to 0} \sum_i \sum_j f(x_i, y_j)\,\Delta x\,\Delta y$$

where (x_i, y_j) are points in the rectangles of area $\Delta x \Delta y$ that make up the region R.



R can be subdivided into infinitesimal rectangles of area $dA = dy\,dx$ or $dA = dx\,dy$



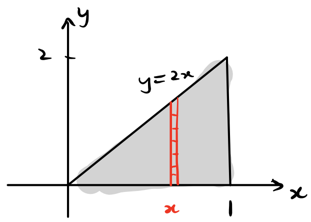
We can think of $f(x,y)\,dA$ as the volume of a rectangular prism of height $f(x,y)$ and $\iint_R f(x,y)\,dA$ as the signed volume between the graph of $f(x,y)$ and the region R in the xy-plane.

Remark When $f(x,y) = 1$, $\iint_R f(x,y)\,dA = \text{area}(R)$

Example Set up and compute the double integral

$\iint_R f(x,y)\,dA$, where $f(x,y)=1$ and R is shown below

① using $dA = dy\,dx$.



For each x, with $0 \le x \le 1$, there is a vertical strip, and this vertical strip is made up of rectangles with area dA for y-values such that

$$0 \le y \le 2x$$

$$\iint_R f(x,y)\,dA = \int_{x=0}^{x=1} \int_{y=0}^{y=2x} 1\,dy\,dx$$

$$= \underbrace{\int_{x=0}^{x=1}}_{\text{summation over each vertical strip}} \left(\underbrace{\int_{y=0}^{y=2x}}_{\text{summation along a fixed vertical strip}} 1\,dy \right) dx$$

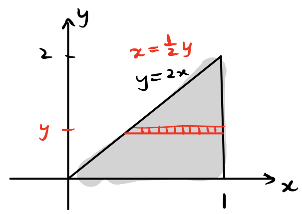
$$= \int_0^1 \left(y \Big|_{y=0}^{y=2x} \right) dx$$

$$= \int_0^1 2x\,dx$$

$$= x^2 \Big|_0^1$$

$$= 1$$

(2) using dA



For each y, with $0 \le y \le 2$, there is a horizontal strip, and this horizontal strip is made up of rectangles with area dA for x-values such that

$$\frac{1}{2}y \le x \le 1$$

$$\iint_R f(x,y)\,dA = \int_{y=0}^{y=2} \int_{x=\frac{1}{2}y}^{x=1} 1\,dx\,dy$$

$$= \int_0^2 \left(\int_{\frac{1}{2}y}^{1} 1\,dx \right) dy$$

$$= \int_0^2 \left(x \Big|_{\frac{1}{2}y}^{1} \right) dy$$

$$= \int_0^2 \left(1 - \frac{1}{2}y \right) dy$$

$$= y - \frac{1}{4}y^2 \Big|_0^2$$

$$= 2 - \frac{1}{4}(2)^2 = 1$$

Remarks When setting up double integrals

(1) the "outside" limits of integration must be constants and "inside" limits might depend on the other variable

(2) the function $f(x,y)$ has no influence on limits

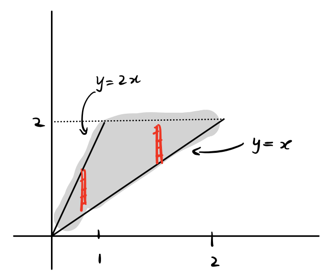
Example Let R be the region between $y=x$ and $y=2x$ for y-values between $y=0$ and $y=2$.

Set up $\iint_R f(x,y)\,dA$ in two ways:

① with $dA = dy\,dx$

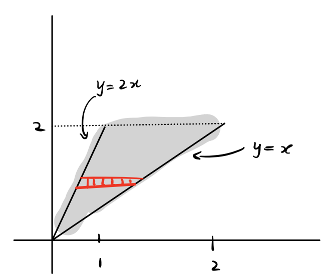
② with $dA = dx\,dy$

① $$\int_{x=0}^{x=1}\int_{y=x}^{y=2x} f(x,y)\,dy\,dx + \int_{x=1}^{x=2}\int_{y=x}^{y=2} f(x,y)\,dy\,dx$$



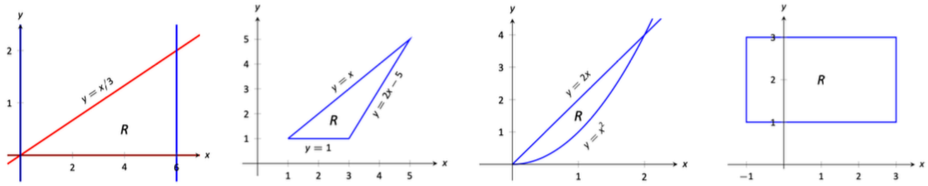
Region must be split into pieces!

② $$\int_{y=0}^{y=2}\int_{x=\frac{1}{2}y}^{x=y} f(x,y)\,dx\,dy$$



no need to split!

Problem 1. Let $f(x,y)$ be a given function. For each region R below, set up the double integral $\iint_R f(x,y)\,dA$ in two ways: using $dA = dydx$ and $dA = dxdy$.



(a)

$$\int_{x=0}^{x=6} \int_{y=0}^{y=x/3} f(x,y)\,dy\,dx$$

$$\int_{y=0}^{y=2} \int_{x=3y}^{x=6} f(x,y)\,dx\,dy$$

(b)

$$\int_{x=1}^{x=3} \int_{y=1}^{y=x} f(x,y)\,dy\,dx + \int_{x=3}^{x=5} \int_{y=2x-5}^{y=x} f(x,y)\,dy\,dx$$

$$\int_{y=1}^{y=5} \int_{x=y}^{x=\frac{1}{2}(y+5)} f(x,y)\,dx\,dy$$

(c)

$$\int_{x=0}^{x=2} \int_{y=x^2}^{y=2x} f(x,y)\,dy\,dx$$

$$\int_{y=0}^{y=4} \int_{x=\frac{1}{2}y}^{x=\sqrt{y}} f(x,y)\,dx\,dy$$

(d)

$$\int_{x=-1}^{x=3} \int_{y=1}^{y=3} f(x,y)\,dy\,dx$$

$$\int_{y=1}^{y=3} \int_{x=-1}^{x=3} f(x,y)\,dx\,dy$$

Problem 2. Each double integral below represents the area of a region R in the xy-plane. Sketch R and then set up the integral again with the order of integration reversed.

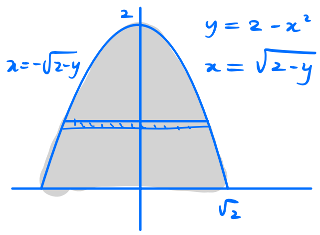
a. $\int_{-\sqrt{2}}^{\sqrt{2}} \int_0^{2-x^2} 1\, dy dx$

b. $\int_0^1 \int_x^{2x} 1\, dy dx$

c. $\int_0^{10} \int_{-10+y}^{10-y} 1\, dx dy$

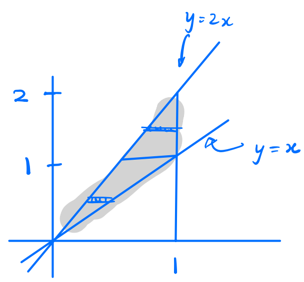
d. $\int_{-1}^{1} \int_{y^2}^{1} 1\, dx dy$

(a)



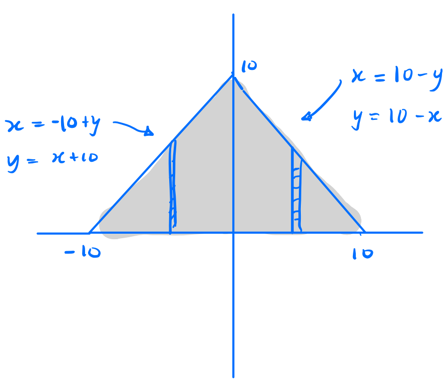
$$\int_{y=0}^{y=2} \int_{x=-\sqrt{2-y}}^{x=\sqrt{2-y}} 1\, dx dy$$

(b)



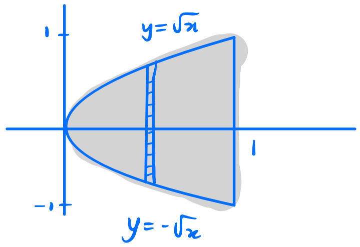
$$\int_{y=0}^{y=1} \int_{x=y/2}^{x=y} 1 dx dy + \int_{y=1}^{y=2} \int_{x=y/2}^{x=1} 1 dx dy$$

(c)



$$\int_{x=-10}^{x=0} \int_{y=0}^{y=x+10} 1 dy dx + \int_{x=0}^{x=10} \int_{y=0}^{10-x} 1 dy dx$$

(d)



$$\int_{x=0}^{x=1} \int_{y=-\sqrt{x}}^{y=\sqrt{x}} 1\, dy\, dx$$

Problem 3. Compute the following double integrals.

a. $\int_0^3 \int_0^1 (5x^3 + 3xy - 4y^2)\, dydx$

b. $\int_0^\pi \int_0^{\pi/2} (x^2 \sin y + \cos x)\, dxdy$

c. $\int_0^1 \int_{x^2}^{x} (x + y)\, dydx$

d. $\int_0^1 \int_0^y e^x\, dxdy$

(a)

$$\int_0^3 \left(5x^3 y + \frac{3}{2} x y^2 - \frac{4}{3} y^3 \Big|_0^1 \right) dx$$

$$= \int_0^3 \left(5x^3 + \frac{3}{2} x - \frac{4}{3} \right) dx$$

$$= \frac{5}{4} x^4 + \frac{3}{4} x^2 - \frac{4}{3} x \Big|_0^3$$

$$= \frac{5}{4} (3)^4 + \frac{3}{4} (3)^2 - 4$$

$$= 104$$

(b)

$$\int_0^\pi \left(\frac{1}{3} x^3 \sin y + \sin x \Big|_0^{\pi/2} \right) dy$$

$$= \int_0^\pi \left(\frac{\pi^3}{24} \sin y + 1 \right) dy$$

$$= -\frac{\pi^3}{24} \cos y + y \Big|_0^\pi$$

$$= \frac{\pi^3}{12} + \pi$$

(c)

$$\int_0^1 \left(xy + \frac{1}{2} y^2 \Big|_{x^2}^{x} \right) dx$$

$$= \int_0^1 \left(\left(x^2 + \frac{1}{2} x^2\right) - \left(x^3 + \frac{1}{2} x^4\right) \right) dx$$

$$= \int_0^1 \left(\frac{3}{2} x^2 - x^3 - \frac{1}{2} x^4 \right) dx$$

$$= \frac{1}{2} x^3 - \frac{1}{4} x^4 - \frac{1}{10} x^5 \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{4} - \frac{1}{10}$$

$$= \frac{3}{20}$$

(d)

$$\int_0^1 \left(e^x \Big|_0^y \right) dy$$

$$= \int_0^1 (e^y - 1)\, dy$$

$$= e^y - y \Big|_0^1$$

$$= e - 2$$