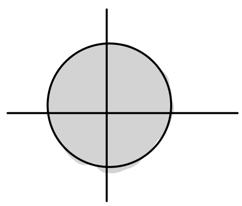
13.3 Polar Integrals

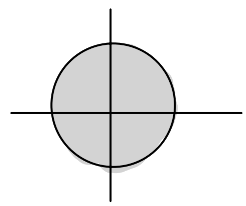
Goal Compute $\iint_R f(x,y)\,dA$ when R is a region in the xy-plane that is easier to describe in polar coordinates than Cartesian coordinates.



unit disk in Cartesian:

$-1 \le x \le 1$

$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$



unit disk in polar:

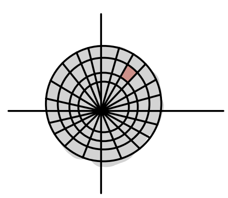
$0 \le \theta \le 2\pi$

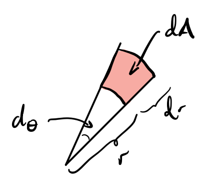
$0 \le r \le 1$

constant bounds are nicer to work with in integrals

Warning In Cartesian coordinates dA is $dy\,dx$ or $dx\,dy$

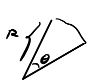
In polar coordinates dA is $r\,dr\,d\theta$.





area of disk with radius R $= \pi R^2 = \frac{1}{2}(2\pi)R^2$

area of wedge of disk with angle θ $= \frac{1}{2}\theta R^2$

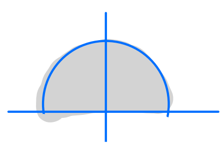


$$\text{area } dA = \tfrac{1}{2}(r+dr)^2\,d\theta - \tfrac{1}{2}r^2\,d\theta$$

$$= \tfrac{1}{2}(r^2 + 2r\,dr + dr^2)\,d\theta - \tfrac{1}{2}r^2\,d\theta$$

$$\approx r\,dr\,d\theta$$ (since dr^2 is order of magnitude too small to count)

<u>Example</u> Set up $\iint_R e^{-(x^2+y^2)}\,dA$ when R is the upper half of the unit disk. Use polar coordinates and then compare with Cartesian coordinates.



$0 \le r \le 1$

$0 \le \theta \le \pi$

$$\int_{x=-1}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} e^{-(x^2+y^2)}\,dy\,dx$$ (impossible to do by hand)

$$\int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=1} e^{-r^2} \cdot r\,dr\,d\theta$$

$u = -r^2$

$du = -2r\,dr$

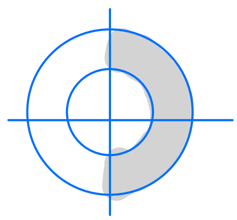
$-\frac{1}{2}\,du = r\,dr$

$$= -\frac{1}{2}\int_0^{\pi}\int_0^{-1} e^u\,du\,d\theta$$

$$= \frac{-1}{2}\int_0^{\pi}\left(e^u\Big|_0^{-1}\right)d\theta$$

$$= \frac{1}{2}\int_0^{\pi}(1-e^{-1})\,d\theta = \frac{\pi}{2}(1-e^{-1})$$

Example Set up $\displaystyle\iint_R \frac{1}{(x^2+y^2)^{3/2}}\, dA$ when R is the right half of the annulus with inner and outer radiuses of 1 and 2. Use polar coordinates and compare with Cartesian.



$1 \le r \le 2$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \int_{r=1}^{r=2} \frac{1}{(r^2)^{3/2}}\, r\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_1^2 r^{-2}\, dr\, d\theta = \frac{\pi}{4}$$

$$\int_{x=0}^{x=1} \int_{y=\sqrt{1-x^2}}^{y=\sqrt{4-x^2}} (x^2+y^2)^{-3/2}\, dy\, dx$$

$$+ \int_{x=0}^{x=1} \int_{y=-\sqrt{4-x^2}}^{y=-\sqrt{1-x^2}} (x^2+y^2)^{-3/2}\, dy\, dx$$

$$+ \int_{x=1}^{x=2} \int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} (x^2+y^2)^{-3/2}\, dy\, dx$$

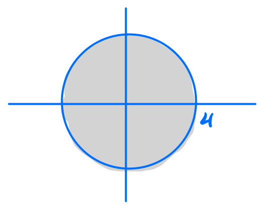
Problem 1. Set up both polar and Cartesian integrals for $\iint f(x,y)\,dA$ where f and R are given as follows.

a. $f(x,y) = 3 + 2x - 4y$, R is the disk of radius 4.

b. $f(x,y) = (x^2+y^2)^{-5/2}$, R is the left half the annulus with inner and outer radiuses 2 and 5.

c. $f(x,y) = e^{-(x^2+y^2)}$, R is the portion of the disk of radius 6 in the second and fourth quadrants.

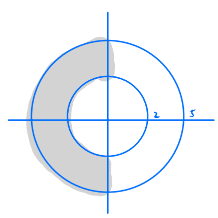
(a)



$$\int_0^{2\pi}\int_0^4 (3 + 2r\cos\theta - 4r\sin\theta)\, r\,dr\,d\theta$$

$$\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (3 + 2x - 4y)\,dy\,dx$$

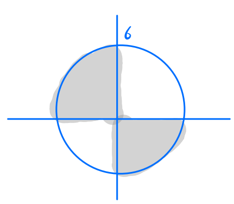
(b)



$$\int_{\pi/2}^{3\pi/2}\int_2^5 (r^2)^{-5/2}\, r\,dr\,d\theta$$

$$\int_{-5}^{-2}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} (x^2+y^2)^{-5/2}\,dy\,dx + \int_{-2}^{0}\int_{\sqrt{4-x^2}}^{\sqrt{25-x^2}} (x^2+y^2)^{-5/2}\,dy\,dx + \int_{-2}^{0}\int_{-\sqrt{25-x^2}}^{\sqrt{4-x^2}} (x^2+y^2)^{-5/2}\,dy\,dx$$

(c)



$$\int_{\pi/2}^{\pi}\int_0^6 e^{-r^2}\, r\,dr\,d\theta + \int_{\frac{3\pi}{2}}^{2\pi}\int_0^6 e^{-r^2}\, r\,dr\,d\theta$$

$$\int_{-6}^{0}\int_0^{\sqrt{36-x^2}} e^{-(x^2+y^2)}\,dy\,dx + \int_0^6\int_{-\sqrt{36-x^2}}^{0} e^{-(x^2+y^2)}\,dy\,dx$$

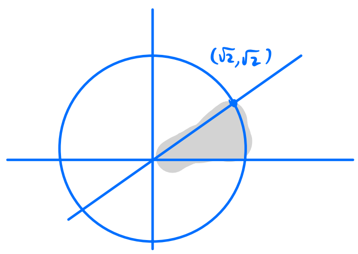
Problem 2. Rewrite the following integrals in polar coordinates:

a. $\int_0^{\sqrt{2}/2} \int_y^{\sqrt{1-y^2}} (x+y)\, dxdy$

b. $\int_{-\sqrt{2}/2}^{0} \int_{-x}^{\sqrt{1-x^2}} (x^2+y^2)\, dydx$

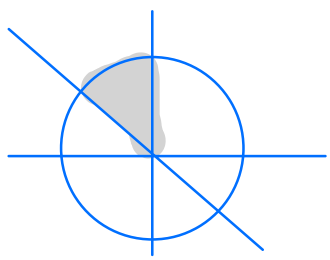
c. $\int_{-1}^{-\sqrt{2}/2} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (3x-y)\, dydx + \int_{-\sqrt{2}/2}^{0} \int_{x}^{\sqrt{1-x^2}} (3x-y)\, dydx$

a



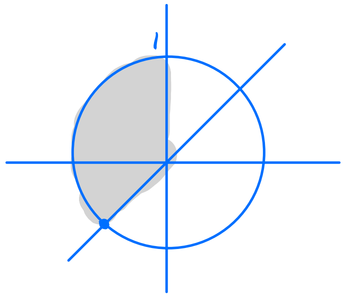
$$\int_0^{\pi/4} \int_0^1 (r\cos\theta + r\sin\theta)\, r\, dr\, d\theta$$

b



$$\int_{\pi/2}^{3\pi/4} \int_0^1 r^3\, dr\, d\theta$$

c



$$\int_{\pi/2}^{5\pi/4} \int_0^1 (3r\cos\theta - r\sin\theta)\, r\, dr\, d\theta$$