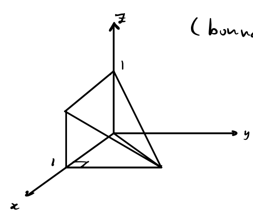
13.6 More triple integrals

<u>Goal</u> Set up triple integrals using other orders of integration (not necessarily using z for inside limits of integration)

<u>Example</u> Let D be the solid region shown below

(bounded by $y=0, z=0, x=1, x=y, z=1-y$)



top surface: $z=1-y$

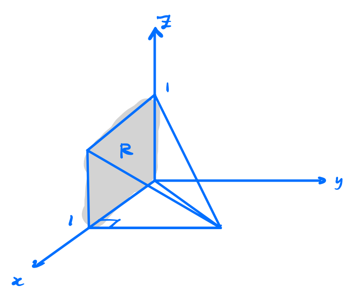
bottom: $z=0$

Set up the triple integral $\iiint\limits_D f(x,y,z)\,dV$ using

① $dV = dy\,dz\,dx$

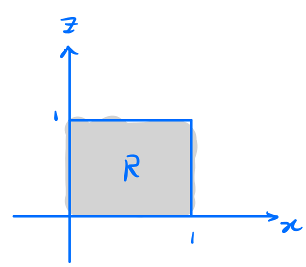
② $dV = dx\,dz\,dy$

①



left surface: $y=0$

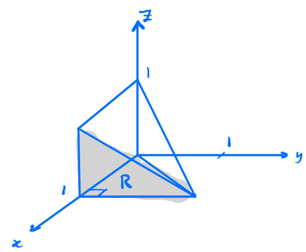
right surface: $y=1-z$



collapse onto xz-plane

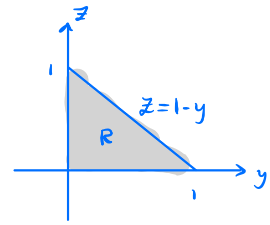
$$\underbrace{\int_{x=0}^{x=1}\int_{z=0}^{z=1}}_{\text{double integral over } R}\ \underbrace{\int_{y=0}^{y=1-z}}_{\text{left, right surfaces}} f(x,y,z)\,dy\,dz\,dx$$

②



back surface : $x = y$

front surface : $x = 1$

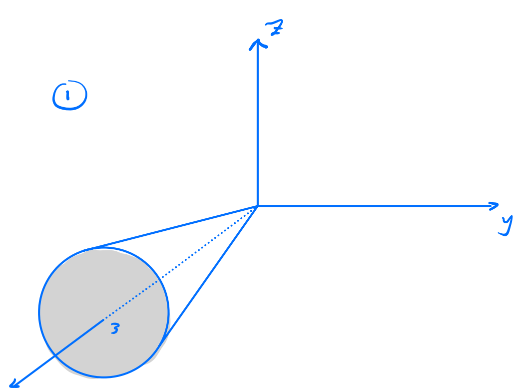


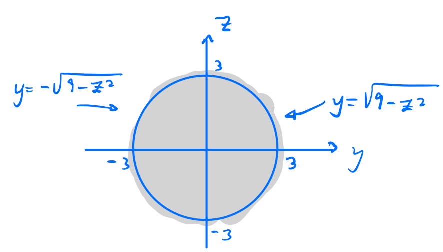
Collapse onto yz-plane

$$\underbrace{\int_{y=0}^{y=1} \int_{z=0}^{z=1-y}}_{\text{double integral over } R} \underbrace{\int_{x=y}^{x=1}}_{\text{front and back surfaces}} f(x,y,z)\, dz\, dy\, dx$$

Example Consider D bounded by $x = \sqrt{y^2+z^2}$ and $x=3$. Set up a triple integral for volume of D using ① $dV = dx\,dy\,dz$ and ② $dV = dz\,dy\,dx$

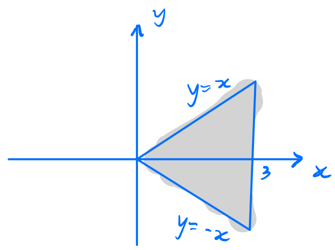
①





$$\int_{-3}^{3} \int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} \int_{\sqrt{y^2+z^2}}^{3} 1\, dx\, dy\, dz$$

(2)



$$\int_0^3 \int_{-x}^{x} \int_{-\sqrt{y^2-x^2}}^{\sqrt{y^2-x^2}} 1 \, dz \, dy \, dx$$

Guide to triple integrals

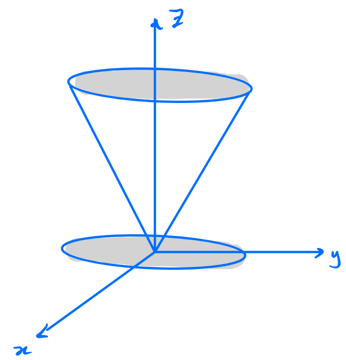
(1) Sketch D

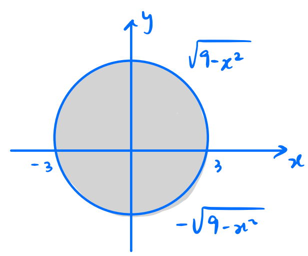
(2) Decide on variable for inner limits

(3) Collapse D onto region R in plane for outer, middle variables. Sketch R.

(4) Outer and middle limits are double integral over R, inner limits are bounding surfaces of D

Problem 1. Let D be the solid bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by the plane $z = 3$. Make a sketch of D and set up the triple integral $\iiint_D f(x,y,z)\,dV$ in two ways: using (1) $dV = dz dy dx$ and (2) $dV = dx dy dz$.



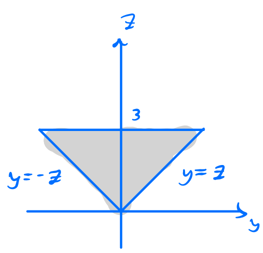


collapse onto xy-plane

intersection :

$$\begin{cases} z = 3 \\ z = \sqrt{x^2+y^2} \end{cases} \Rightarrow 9 = x^2+y^2$$

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{\sqrt{x^2+y^2}}^{3} f(x,y,z)\,dz\,dy\,dx$$

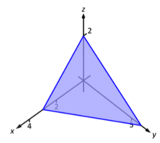


collapse onto yz-plane

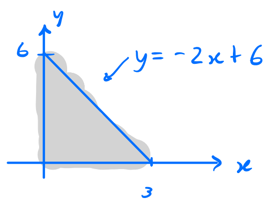
$$\int_{z=0}^{z=3}\int_{y=-z}^{y=z}\int_{x=-\sqrt{z^2-y^2}}^{x=\sqrt{z^2-y^2}} f(x,y,z)\,dx\,dy\,dz$$

(back and front surfaces found by solving $z = \sqrt{x^2+y^2}$ for x)

Problem 2. Let D be the region bounded by $x = 0, y = 0, z = 0$, and $z = 2 - y/3 - 2x/3$ (shown below). Set up the triple integral $\iiint_D dV$ using the following choices of dV: (1) $dz\,dy\,dx$, (2) $dx\,dy\,dz$, and (3) $dx\,dz\,dy$.

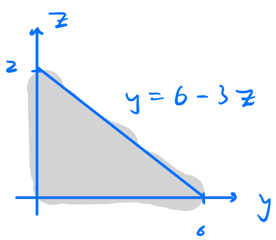


(1)



$$\int_0^3 \int_0^{-2x+6} \int_0^{2-y/3-2x/3} 1\,dz\,dy\,dx$$

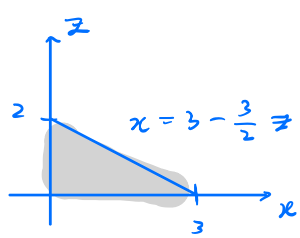
(2)



$$z = 2 - y/3 - 2x/3$$
$$\Rightarrow 3z = 6 - y - 2x$$
$$\Rightarrow x = 3 - \tfrac{1}{2}y - \tfrac{3}{2}z$$

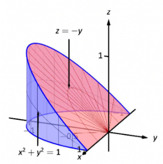
$$\int_0^2 \int_0^{6-3z} \int_0^{3-\frac{1}{2}y-\frac{3}{2}z} 1\,dx\,dy\,dz$$

(3)

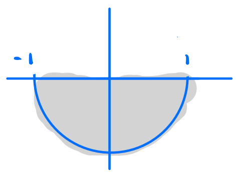


$$\int_0^2 \int_0^{3-\frac{3}{2}z} \int_0^{6-2x-3z} 1\,dy\,dx\,dz$$

Problem 3. Let D be the region shown below. Set up the triple integral $\iiint_D dV$ using the following choices of dV: (1) $dzdydx$, (2) $dxdydz$, and (3) $dydxdz$.

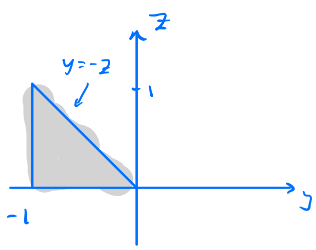


1



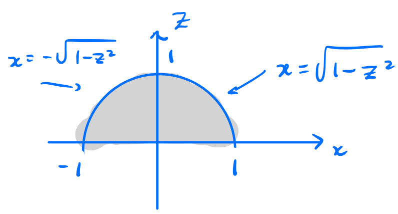
$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0}\int_{0}^{-y} 1\,dz\,dy\,dx$$

2



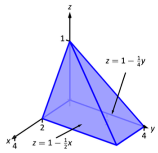
$$\int_{0}^{1}\int_{-1}^{-z}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 1\,dx\,dy\,dz$$

3

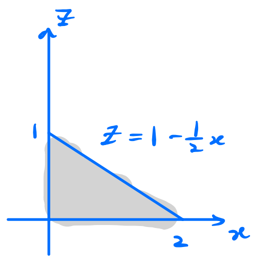


$$\int_{0}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{-\sqrt{1-x^2}}^{-z} 1\,dy\,dx\,dz$$

Problem 4. Let D be the region shown below. Set up the triple integral $\iiint_D dV$ using the following choices of dV: (1) $dydzdx$ and (2) $dzdydx$. *Hint: the second setup is trickier than previous problems; you'll need to write it as a sum of two triple integrals.*

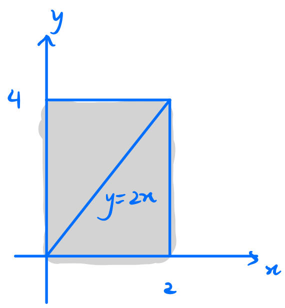


(1)



$$\int_{x=0}^{x=2}\int_{z=0}^{z=1-\frac{1}{2}x}\int_{y=0}^{y=4(1-z)} 1\,dy\,dz\,dx$$

(2)



$$\int_{x=0}^{x=2}\int_{y=2x}^{y=4}\int_{z=0}^{z=1-\frac{1}{4}y} dz\,dy\,dx$$

$$+\int_{x=0}^{x=2}\int_{y=0}^{y=2x}\int_{z=0}^{z=1-\frac{1}{2}x} dz\,dy\,dx$$