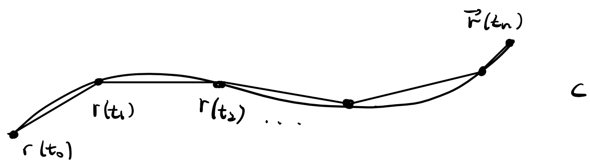
14.2 Line Integrals of Scalar Fields

Def Let C be a curve in $\mathbb{R}^2$ or $\mathbb{R}^3$ given by the vector-valued function $\vec{r}(t)$, $a \le t \le b$.

We often call C a parametrized curve and $\vec{r}(t)$ is called the parametrization.



$$\text{Length of } C \approx \sum_{i=1}^{n} \|\vec{r}(t_i) - \vec{r}(t_{i-1})\|$$

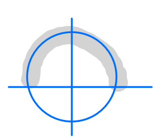
$$= \sum_{i=1}^{n} \frac{\|\vec{r}(t_i) - \vec{r}(t_{i-1})\|}{t_i - t_{i-1}} \underbrace{(t_i - t_{i-1})}_{= \Delta t}$$

$$\longrightarrow \int_a^b \|\vec{r}\,'(t)\|\, dt \quad \text{as } \Delta t \to 0$$

$$\boxed{\text{Arclength} = \int_C 1\, ds = \int_a^b \|\vec{r}\,'(t)\|\, dt}$$

Notation ds is used to represent an infinitesimal arclength segment of the curve C, it's defined as $ds = \|\vec{r}\,'(t)\|\, dt$ once a parametrization is given.

Example Set up and compute an arclength integral when C is the top half of the unit circle.



$\vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le \pi$

$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$, $\|\vec{r}\,'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$

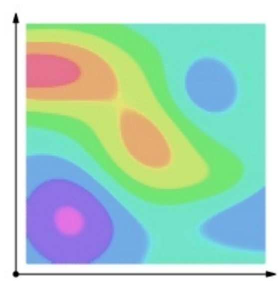
$$\int_C ds = \int_0^{\pi} \|\vec{r}\,'(t)\|\, dt = \int_0^{\pi} dt = \pi$$

We'd like to generalize this idea and compute the integral of a function $f: D \to \mathbb{R}$ ($D \subseteq \mathbb{R}^2$ or $\mathbb{R}^3$) over a curve C:

$$\boxed{\int_C f\,ds = \int_a^b f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt}$$

This called a <u>line integral</u> over <u>the scalar field</u> f

<u>Notation</u> When C is a <u>closed</u> curve (it starts and ends at the same point), the notation $\oint_C f\,ds$ sometimes gets used.



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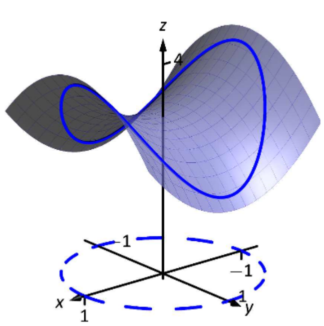
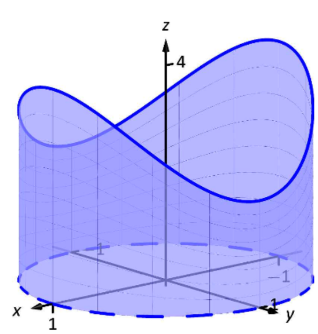
<u>Example</u> Let $f(x,y) = x^2 - y^2 + 3$ and let C be the unit circle $x^2 + y^2 = 1$. Compute $\int_C f\,ds$.





Let $\vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le 2\pi$ parametrize C.

Then $\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$ and $\|\vec{r}\,'(t)\| = 1$.

Then $$\oint_C f\,ds = \int_0^{2\pi} f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt$$

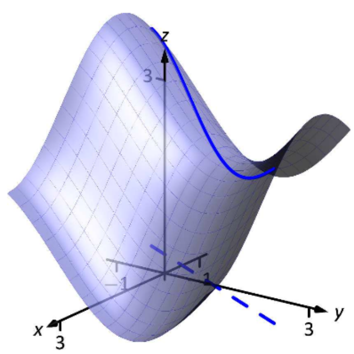
$$= \int_0^{2\pi} \left(\cos^2 t - \sin^2 t + 3\right) dt$$

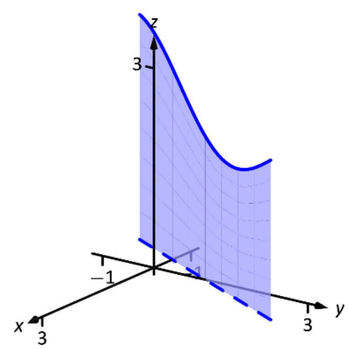
$$= 6\pi$$ (using software, eg. Wolfram Alpha)

Example Let $f(x,y) = \cos x + \sin y + 2$ and let

C be the segment of the line $y = 2x+1$ where $-1 \le x \le 1$.

Compute $\int_C f\,ds$.





Let $\vec{r}(t) = \langle t, 2t+1 \rangle$, $-1 \le t \le 1$. Then

$\vec{r}\,'(t) = \langle 1, 2 \rangle$, $\|\vec{r}\,'(t)\| = \sqrt{5}$.

$$\int_C f\,ds = \int_{-1}^{1} \left(\cos t + \sin(2t+1) + 2\right)\sqrt{5}\,dt$$

$$= 4\sqrt{5} + \sqrt{5}\int_{-1}^{1} \sin(2t+1)\,dt$$ $u = 2t+1$, $du = 2\,dt$

$$= 4\sqrt{5} + \frac{\sqrt{5}}{2}\int_{-1}^{3} \sin u\,du$$

$$= 4\sqrt{5} + \frac{\sqrt{5}}{2}\left(\cos(1) - \cos(3)\right) \approx 14.418$$

Example Find area under $f(x,y) = 1 - \cos x \sin y$

and above the parabola $y = x^2$, $-1 \le x \le 1$, in xy-plane.

Let $\vec{r}(t) = \langle t, t^2 \rangle$, $-1 \le t \le 1$. Then $\vec{r}\,'(t) = \langle 1, 2t \rangle$

and $\|\vec{r}\,'(t)\| = \sqrt{1+4t^2}$. So

$$\int_C f\,ds = \int_{-1}^{1} (1 - \cos t \sin t^2)\sqrt{1+4t^2}\,dt$$

$$\approx 2.17 \quad \text{(must use software)}$$

Problem 1. Set up the line integral $\int_C 1\,ds$ for the arclength of each given curve.

a. C is the portion of $y = x^3$ from $(-1,-1)$ to $(1,1)$.

b. C is the portion of $x = -1 + y^2$ from $(0,-1)$ to $(0,1)$.

c. C is the upper half of the ellipse $\mathbf{r}(t) = \langle 2\cos t, \sin t \rangle$.

a) $\vec{r}(t) = \langle t, t^3 \rangle$, $-1 \le t \le 1$

$\vec{r}\,'(t) = \langle 1, 3t^2 \rangle$, $\|\vec{r}\,'(t)\| = \sqrt{1+9t^4}$

$$\int_{-1}^{1} \sqrt{1+9t^4}\,dt$$

b) $\vec{r}(t) = \langle -1+t^2, t \rangle$, $-1 \le t \le 1$

$\vec{r}\,'(t) = \langle 2t, 1 \rangle$, $\|\vec{r}\,'(t)\| = \sqrt{4t^2+1}$

$$\int_{-1}^{1} \sqrt{1+4t^2}\,dt$$

c) $\vec{r}\,'(t) = \langle -2\sin t, \cos t \rangle$, $\|\vec{r}\,'(t)\| = \sqrt{4\sin^2 t + \cos^2 t}$

$= \sqrt{3\sin^2 t + 1}$

$$\int_0^{\pi} \sqrt{3\sin^2 t + 1}\,dt$$

Problem 2. Set up the line integral $\int_C f\,ds$ for each given plane curve C and surface $f(x, y)$. Use software like Wolfram Alpha to compute the integral.

a. C is the parabola $x = y^2$ oriented from $(4, -2)$ to $(4, 2)$; $f(x, y) = \sin x \cos y$

b. C is the line segment joining $(-2, -1)$ and $(1, 2)$; $f(x, y) = x^2 + y^2 + 2$

c. C is the right half of the unit circle along with the line segment from $(0, 1)$ to $(0, -1)$; $f(x, y) = x + y$

d. C is the line segments from $(0, 1)$ to $(1, 1)$ and from $(1, 1)$ to $(1, 0)$; $f(x, y) = x + y^2$

(a) $\vec{r}(t) = \langle t^2, t\rangle, \quad -2 \le t \le 2.$

$\vec{r}\,'(t) = \langle 2t, 1\rangle, \quad \|\vec{r}\,'(t)\| = \sqrt{4t^2+1}$

$$\int_C f\,ds = \int_{-2}^{2} \sin t^2 \cos t \sqrt{4t^2+1}\,dt$$

(b) $\vec{r}(t) = \langle -2, -1\rangle + t\langle 3, 3\rangle$

$= \langle -2+3t, -1+3t\rangle, \quad 0 \le t \le 1$

$\vec{r}\,'(t) = \langle 3, 3\rangle, \quad \|\vec{r}\,'(t)\| = \sqrt{18}$

$$\int_C f\,ds = \int_0^1 \left((-2+3t)^2 + (-1+3t)^2 + 2\right)\sqrt{18}\,dt$$

(c) $\vec{r}_1(t) = \langle \cos t, \sin t\rangle, \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}, \quad \vec{r}_1\,'(t) = \langle -\sin t, \cos t\rangle, \quad \|\vec{r}\,'(t)\| = 1$

$\vec{r}_2(t) = \langle 0, t\rangle, \quad -1 \le t \le 1, \quad \vec{r}_2\,'(t) = \langle 0, 1\rangle, \quad \|\vec{r}\,'(t)\| = 1$

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos t + \sin t)\,dt + \int_{-1}^{1} t\,dt$$

(d) $\vec{r}_1(t) = \langle 0, 1\rangle + t\langle 1, 0\rangle = \langle t, 1\rangle, \quad 0 \le t \le 1$

$\vec{r}_1\,'(t) = \langle 1, 0\rangle, \quad \|\vec{r}_1\,'(t)\| = 1$

$\vec{r}_2(t) = \langle 1, 1\rangle + t\langle 0, -1\rangle = \langle 1, -t\rangle, \quad 0 \le t \le 1$

$\vec{r}_2\,'(t) = \langle 0, -1\rangle, \quad \|\vec{r}_2\,'(t)\| = 1$

$$\int_C f\,ds = \int_{C_1} f\,ds + \int_{C_2} f\,ds$$

$$= \int_0^1 (t+1)\,dt + \int_0^1 (1+t^2)\,dt$$

Problem 3. Let C be the piecewise defined closed curve in the shape of a square with vertices at $(\pm 1, \pm 1)$. Let C_L, C_R, C_T, C_B denote the four line segments that form C (left, right, top, bottom respectively). Give the sign (positive, negative, or zero) of each of the following line integrals.

a. $\int_{C_R} x\,ds$ +

b. $\int_{C_L} x\,ds$ −

c. $\int_{C_T} x\,ds$ 0

d. $\int_{C_B} x\,ds$ 0

e. $\int_C x\,ds$ 0

f. $\int_C (x^2 + y^2)\,ds$ +