14.2 Vector Fields

A <u>vector field</u> is a function $F: \mathbb{R}^n \to \mathbb{R}^n$, $n = 2$ or 3, whose outputs are vectors:

$$F(x,y) = \langle M(x,y), N(x,y) \rangle \quad (n=2)$$

$$F(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z) \rangle \quad (n=3).$$

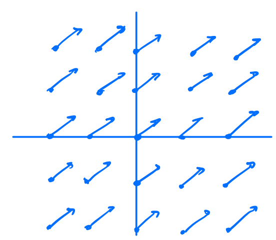
The idea is to associate a vector to every point in the plane or in space. They're meant to model things like fluid velocity or gravitational force or electromagnetic force.

<u>Examples</u> Sketch the following vector fields.

① $\vec{F}(x,y) = \langle 2, 2 \rangle$

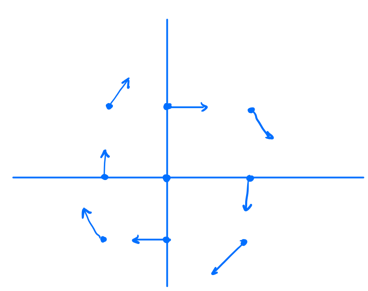
at every (x,y) we plot the vector $\langle 2,2 \rangle$

this is a "constant" vector field, no dependence on x and y



② $\vec{F}(x,y) = \langle y, -x \rangle$

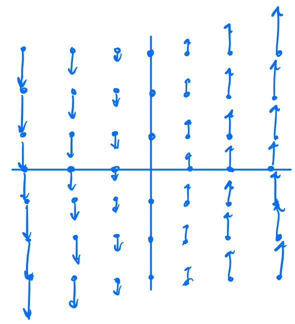
(x,y)	$\vec{F}(x,y)$
$(0,0)$	$\langle 0,0 \rangle$
$(0,1)$	$\langle 1,0 \rangle$
$(0,-1)$	$\langle -1,0 \rangle$
$(1,0)$	$\langle 0,-1 \rangle$
$(1,1)$	$\langle 1,-1 \rangle$
$(1,-1)$	$\langle -1,-1 \rangle$
$(-1,0)$	$\langle 0,1 \rangle$
$(-1,1)$	$\langle 1,1 \rangle$
$(-1,-1)$	$\langle -1,1 \rangle$



(we'll draw the vectors not to scale)

③ $F(x,y) = \langle 0, x \rangle$

(x,y)	$F(x,y)$	
$(0,c)$	$\langle 0,0 \rangle$	for any c
$(1,c)$	$\langle 0,1 \rangle$	for any c
$(-1,c)$	$\langle 0,-1 \rangle$	for any c



Flow lines

Suppose the vectors in a vector field $\vec{F}$ represent the velocity vectors of a particle moving in $\mathbb{R}^2$ or $\mathbb{R}^3$

A flow line is a parametrized curve $\vec{r}(t)$ such that $\vec{r}\,'(t) = \vec{F}(\vec{r}(t))$.

It represents the position $\vec{r}(t)$ of the particle at time t.

Example If $\vec{F}(x,y) = \langle y, -x \rangle$, what equations does a flow line $\vec{r}(t) = \langle x(t), y(t) \rangle$ satisfy?

$$\vec{r}\,'(t) = \vec{F}(\vec{r}(t))$$

means $\langle x'(t), y'(t) \rangle = \langle y(t), -x(t) \rangle$

means $\begin{cases} x'(t) = y(t) \\ y'(t) = -x(t) \end{cases}$

Check that $\vec{r}_1(t) = \langle \cos(-t), \sin(-t) \rangle$

or $\vec{r}_2(t) = \langle \sin t, \cos t \rangle$

are flow lines.

$$\vec{r}_1{}'(t) = \langle \sin(-t), -\cos(-t) \rangle$$

$$\vec{F}(r_1(t)) = \langle \sin(-t), -\cos(-t) \rangle$$

$$\vec{r}_2{}'(t) = \langle \cos t, -\sin t \rangle$$

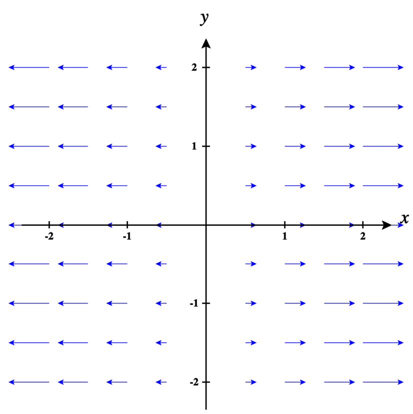
$$\vec{F}(\vec{r}_2(t)) = \langle \cos t, -\sin t \rangle$$

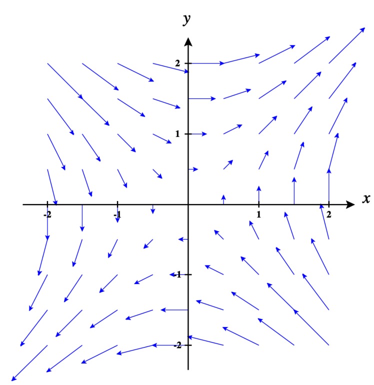
Problem 1. Sketch the following vector fields by hand and then check your work in CalcPlot3d.

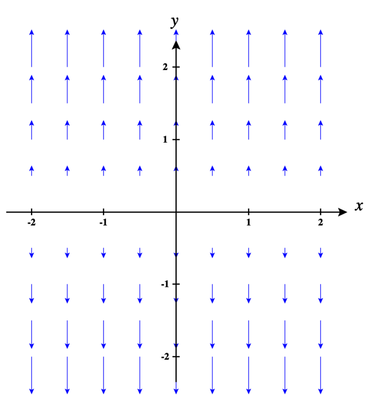
a. $\mathbf{F}(x, y) = \langle x, 0 \rangle$

b. $\mathbf{F}(x, y) = \langle y, x \rangle$

c. $\mathbf{F}(x, y) = \langle 0, y \rangle$





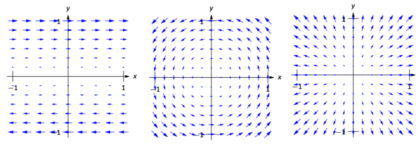


Problem 2. Match each of the following vector fields with their plots shown below and then trace in an example flow line that starts in the second quadrant.

a. $\mathbf{F}(x, y) = \langle x, y\rangle$

b. $\mathbf{F}(x, y) = \langle -y, x\rangle$

c. $\mathbf{F}(x, y) = \langle y, 0\rangle$



I II III

F	plot
a	III
b	II
c	I

Problem 3. Match the three vector fields in the previous problem with the corresponding flow lines.

a. $\mathbf{r}_1(t) = \langle ae^t, be^t\rangle$

b. $\mathbf{r}_2(t) = \langle a\cos t, a\sin t\rangle$

c. $\mathbf{r}_3(t) = \langle at + b, a\rangle$

$\vec{r}(t)$	F
$\vec{r}_1$	a
$\vec{r}_2$	b
$\vec{r}_3$	c

$\vec{r}_1'(t) = \langle ae^t, be^t\rangle$

$F_a(\vec{r}_1(t)) = \langle ae^t, be^t\rangle$

$\vec{r}_2(t) = \langle -a\sin t, a\cos t\rangle$

$\vec{F}_b(\vec{r}_2(t)) = \langle -a\sin t, a\cos t\rangle$

$\vec{r}_3'(t) = \langle a, 0\rangle$

$\vec{F}(\vec{r}_3(t)) = \langle a, 0\rangle$

Problem 4. Pair each contour plot for a function f with the vector field $\mathbf{F}$ so that $\mathbf{F} = \nabla f$. These vector fields are called ***gradient vector fields*** and the functions f are called ***potential functions***.