

10.3 Dot Product

Def Let $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$.

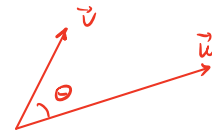
Their dot product is the scalar given by

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2$$

(similar definition for vectors in \mathbb{R}^3).

Theorem Let θ be the angle in $[0, \pi]$ between the vectors \vec{v} and \vec{w} . Then

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos \theta.$$



Example Compute the angle between $\vec{v} = \langle 1, 0, 2 \rangle$

and $\vec{w} = \langle -1, 1, 4 \rangle$.

$$\theta = \cos^{-1} \left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|} \right)$$

$$= \cos^{-1} \left(\frac{-1 + 0 + 8}{\sqrt{5} \cdot \sqrt{18}} \right)$$

$$= \cos^{-1} \left(\frac{7}{\sqrt{90}} \right)$$

Important idea If $\vec{v} \cdot \vec{w} = 0$, then $\theta = \frac{\pi}{2}$

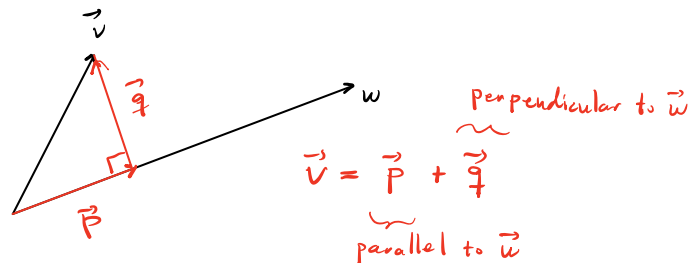
and \vec{v} and \vec{w} are orthogonal (ie perpendicular)

Algebraic properties

- ① $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$
- ② $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
- ③ $c(\vec{v} \cdot \vec{w}) = (c\vec{v}) \cdot \vec{w} = \vec{v} \cdot (c\vec{w})$
- ④ $\vec{0} \cdot \vec{v} = 0$
- ⑤ $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

Orthogonal Projection

Given vector \vec{v} we sometimes want to "decompose" it in terms of another vector \vec{w}

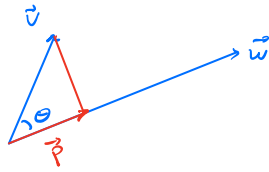


The vector \vec{p} in the sketch above is called the orthogonal projection of \vec{v} onto \vec{w} .

Notation $\text{proj}_{\vec{w}} \vec{v}$

Theorem $\text{proj}_{\vec{w}} \vec{v} = \left(\frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) \vec{w}$

Proof



By trigonometry $\cos \theta = \frac{\|\vec{p}\|}{\|\vec{v}\|}$, so

$$\begin{aligned}\|\vec{p}\| &= \|\vec{v}\| \cos \theta \\ &= \|\vec{v}\| \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|} \quad \text{by dot product angle formula} \\ &= \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|}.\end{aligned}$$

Note $\vec{p} = c\vec{w}$ for some c and so

$$\|c\vec{w}\| = \|\vec{p}\| = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|}$$

which implies $|c| = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2} = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}$ and

so $\vec{p} = \left(\frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) \vec{w}$ (details about abs. value omitted)

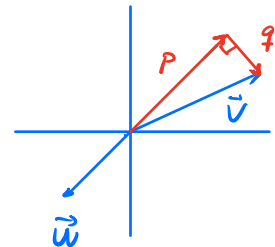
Example Let $\vec{v} = \langle 2, 1 \rangle$ and $\vec{w} = \langle -1, -1 \rangle$

Find vectors \vec{p} and \vec{q} so that $\vec{v} = \vec{p} + \vec{q}$

and \vec{p} and \vec{q} are parallel and perpendicular to \vec{w} .

$$\vec{p} = \text{proj}_{\vec{w}} \vec{v} = \left(\frac{-2-1}{2} \right) \vec{w} = \frac{-3}{2} \langle -1, -1 \rangle = \left\langle \frac{3}{2}, \frac{3}{2} \right\rangle$$

$$\vec{q} = \vec{v} - \vec{p} = \langle 2, 1 \rangle - \left\langle \frac{3}{2}, \frac{3}{2} \right\rangle = \left\langle \frac{1}{2}, -\frac{1}{2} \right\rangle$$



Problem 1. Find the angle between the following vectors. You can leave your answer in terms of \cos^{-1} but make note if any vectors are orthogonal (in which case the angle is $\pi/2$).

- a. $\langle 3, 1 \rangle$ and $\langle -4, 3 \rangle$
- b. $\langle 1, 2 \rangle$ and $\langle 2, -1 \rangle$
- c. $\langle 1, -8, 2 \rangle$ and $\langle 0, 1, 4 \rangle$

$$\textcircled{a} \quad \cos^{-1} \left(\frac{-12+3}{\sqrt{10} \sqrt{25}} \right) = \cos^{-1} \left(\frac{-9}{5\sqrt{10}} \right)$$

$$\textcircled{b} \quad \text{dot product: } 2 + (-2) = 0, \text{ so orthogonal}$$

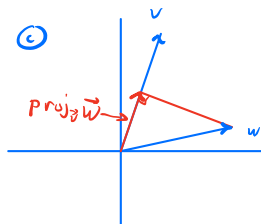
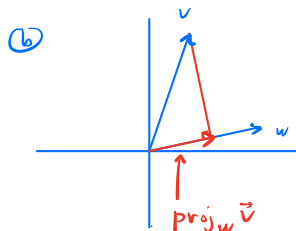
$$\textcircled{c} \quad \text{dot product} = 0 - 8 + 8 = 0, \text{ so orthogonal}$$

Problem 2. Find the orthogonal projection of $\langle 1, 1, 2 \rangle$ onto $\langle 0, 1, 1 \rangle$.

$$\left(\frac{0+1+2}{0^2+1^2+1^2} \right) \langle 0, 1, 1 \rangle = \left\langle 0, \frac{3}{2}, \frac{3}{2} \right\rangle$$

Problem 3. Consider the vectors $\mathbf{v} = \langle 1, 5 \rangle$ and $\mathbf{w} = \langle 4, 1 \rangle$.

- Make a sketch \mathbf{v} and \mathbf{w} on the same set of axes.
- Make a sketch of the $\text{proj}_{\mathbf{w}} \mathbf{v}$.
- Make a sketch of the $\text{proj}_{\mathbf{v}} \mathbf{w}$.
- Find vectors \mathbf{p} and \mathbf{q} so that $\mathbf{v} = \mathbf{p} + \mathbf{q}$ where \mathbf{p} is parallel to \mathbf{w} and \mathbf{q} is perpendicular to \mathbf{w} .
- Find vectors \mathbf{r} and \mathbf{s} so that $\mathbf{w} = \mathbf{r} + \mathbf{s}$ where \mathbf{r} is parallel to \mathbf{v} and \mathbf{s} is perpendicular to \mathbf{v} .



$$\begin{aligned} \text{(d)} \quad \vec{p} &= \text{proj}_{\vec{w}} \vec{v} = \left(\frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) \vec{w} \\ &= \left(\frac{4+5}{17} \right) \langle 4, 1 \rangle = \left\langle \frac{36}{17}, \frac{9}{17} \right\rangle \end{aligned}$$

$$\begin{aligned} \vec{q} &= \vec{v} - \vec{p} = \langle 1, 5 \rangle - \left\langle \frac{36}{17}, \frac{9}{17} \right\rangle \\ &= \left\langle -\frac{19}{17}, \frac{76}{17} \right\rangle \end{aligned}$$

$$\begin{aligned} \text{(e)} \quad \vec{r} &= \text{proj}_{\vec{v}} \vec{w} = \left(\frac{\vec{v} \cdot \vec{w}}{\vec{v} \cdot \vec{v}} \right) \vec{v} \\ &= \left(\frac{9}{26} \right) \langle 1, 5 \rangle = \left\langle \frac{9}{26}, \frac{45}{26} \right\rangle \end{aligned}$$

$$\begin{aligned} \vec{s} &= \vec{w} - \vec{r} = \langle 4, 1 \rangle - \left\langle \frac{9}{26}, \frac{45}{26} \right\rangle \\ &= \left\langle \frac{95}{26}, -\frac{19}{26} \right\rangle \end{aligned}$$