14.3 More Line Integrals

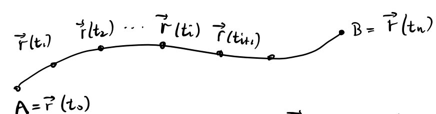
The fundamental theorem of calculus from calculus I, II says $\int_a^b f'(x)\,dx = f(b) - f(a)$

Theorem (Fundamental Theorem of Line Integrals)

If C is a piecewise smooth curve connecting A and B and ∇f is continuous along C then

$$\int_C \nabla f \cdot dr = f(B) - f(A)$$

Informal derivation



$$\Delta \vec{r}_i = \vec{r}(t_{i+1}) - \vec{r}(t_i)$$

$$f(\vec{r}(t_i)) - f(\vec{r}(t_{i-1})) \approx (\text{rate of change of } f)\,\|\Delta \vec{r}_i\|$$

$$= D_{\vec{u}} f(\vec{r}(t_i))\,\|\Delta \vec{r}_i\| \qquad \text{where } \vec{u} = \frac{\Delta \vec{r}_i}{\|\Delta \vec{r}_i\|}$$

$$= \nabla f(\vec{r}(t_i)) \cdot \vec{u}\,\|\Delta \vec{r}_i\|$$

$$= \nabla f(\vec{r}(t_i)) \cdot \Delta \vec{r}_i$$

Therefore $\displaystyle\int_C \nabla f \cdot d\vec{r} \approx \sum_{i=1}^{n} \nabla f(\vec{r}(t_i)) \cdot \Delta \vec{r}_i \quad (\text{as } \|\Delta r\| \to 0)$

$$= \sum_{i=1}^{n} \left(f(\vec{r}(t_i)) - f(\vec{r}(t_{i-1})) \right) = f(B) - f(A)$$

Remark Notice $\int_C \nabla f \cdot d\vec{r}$ only depends on the endpoints.

if $\vec{F}$ represents force, path-independence means work done by $\vec{F}$ only depends on start and end points (like when $\vec{F}$ is force due to gravity)

Def $\vec{F}$ is called path-independent if $\int_C \vec{F} \cdot d\vec{r}$ is the same value for all piecewise smooth curves C from A to B which lie in the domain of $\vec{F}$.

$\vec{F}$ is called conservative if $\vec{F} = \nabla f$ for some function f (called a potential function of $\vec{F}$).

"conservative" comes from "conservation of energy"

Theorem A continuous vector field $\vec{F}$ is path-independent if and only if it is conservative.

Example Let $\vec{F}(x,y) = \langle 3x^2y + 2x, x^3 + 1 \rangle$.

Show that $\vec{F}$ has a potential function by finding one.

Need to find $f : \mathbb{R}^2 \to \mathbb{R}$ so that $\nabla f = \vec{F}$:

$$\begin{cases} f_x = 3x^2y + 2x \\ f_y = x^3 + 1 \end{cases}$$

Note

$$\begin{aligned} f(x,y) &= \int f_y \, dy \\ &= \int (x^3+1) \, dy \\ &= y(x^3+1) + C_1(x) \\ &= x^3y + y + C_1(x) \end{aligned}$$

and

$$\begin{aligned} f(x,y) &= \int f_x \, dx \\ &= \int (3x^2y + 2x) \, dx \\ &= x^3y + x^2 + C_2(y) \end{aligned}$$

To make this work we can choose $C_1(x) = x^2$ and $C_2(y) = y$, so that

$$f(x,y) = x^3y + x^2 + y$$

Example Let $\vec{F}(x,y) = \langle y, x \rangle$ and let C be an oriented curve connecting $(0,1)$ to $(2,3)$. Show $\vec{F}$ is conservative and compute $\int_C \vec{F} \cdot d\vec{r}$.

If $F = \nabla f$ then $f_x = y$ and $f_y = x$.

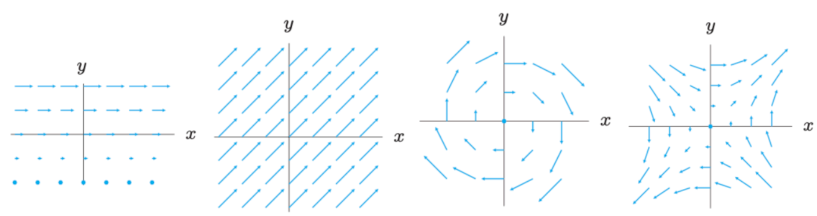
This means
$$f(x,y) = \int f_x \, dx = \int y \, dx$$
$$= xy + C_1(y)$$

and
$$f(x,y) = \int f_y \, dy = \int x \, dy$$
$$= xy + C_2(x)$$

We can let $C_1(y) = C_2(x) = 0$ so that $f(x,y) = xy$.

Therefore
$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r}$$
$$= f(2,3) - f(0,1) = (2)(3) - (0)(1) = 6$$

Problem 1. Let C be the closed curve consisting of a square centered at the origin with vertices at $(\pm 1, \pm 1)$, oriented counter-clockwise. Determine the sign (positive, negative, or zero) of $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for the vector fields $\mathbf{F}$ shown below. Indicate whether the vector field is path-independent.



sign	−	0	−	0
path-independent	no	yes	no	yes

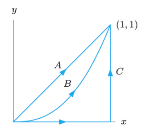
Problem 2. For each of the following vector fields $\mathbf{F}$, find a potential function f of $\mathbf{F}$ and use the Fundamental Theorem of Line Integrals to compute $\int_A \mathbf{F}\cdot d\mathbf{r}$, $\int_B \mathbf{F}\cdot d\mathbf{r}$, and $\int_C \mathbf{F}\cdot d\mathbf{r}$ where A, B, and C are the curves shown below.

a. $\mathbf{F}(x,y) = \langle x, y\rangle$

b. $\mathbf{F}(x,y) = \langle y\cos x, \sin x + y\rangle$

c. $\mathbf{F}(x,y) = \langle 2xy, x^2\rangle$

d. $\mathbf{F}(x,y) = \langle 2xy, x^2 + 8y^3\rangle$



(a) $f_x = x \Rightarrow f(x,y) = \int x\,dx = \frac{1}{2}x^2 + C_1(y)$,

$f_y = y \Rightarrow f(x,y) = \int y\,dy = \frac{1}{2}y^2 + C_2(x)$

$\Rightarrow f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$ (+ any constant)

$$\int_A \vec{F}\cdot d\vec{r} = \int_B \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r} = f(1,1) - f(0,0) = 1$$

(b) $f_x = y\cos x \Rightarrow f(x,y) = \int y\cos x\,dx = y\sin x + C_1(y)$

$f_y = \sin x + y \Rightarrow f(x,y) = \int (\sin x + y)\,dy = y\sin x + \frac{1}{2}y^2 + C_2(x)$

$\Rightarrow C_1(y) = \frac{1}{2}y^2$, $C_2(x) = 0$ (or any constant)

$\Rightarrow f(x,y) = y\sin x + \frac{1}{2}y^2$

$$\int_A \vec{F}\cdot d\vec{r} = \int_B \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r} = f(1,1) - f(0,0) = \sin(1) + \frac{1}{2}$$

(c) $f_x = 2xy \Rightarrow f(x,y) = \int 2xy\,dx = x^2y + C_1(y)$

$f_y = x^2 \Rightarrow f(x,y) = \int x^2\,dy = x^2y + C_2(x)$

$\Rightarrow C_1(y) = C_2(x) = 0$ (or any constant)

$\Rightarrow f(x,y) = x^2y$

$$\int_A \vec{F}\cdot d\vec{r} = \int_B \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r} = f(1,1) - f(0,0) = 1$$

(d) $f_x = 2xy \implies f(x,y) = x^2y + C(y)$

$x^2 + 8y^3 = f_y = x^2 + C'(y)$

$\implies C'(y) = 8y^3$

$\implies C(y) = \int 8y^3\,dy = 2y^4$ (+ any constant)

$f(x,y) = x^2y + 2y^4$

$\int_A \vec{F}\cdot d\vec{r} = \int_B \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r} = f(1,1) - f(0,0) = 3$

Problem 3. Try finding a potential function for $\mathbf{F}(x,y) = \langle -y, x\rangle$. What goes wrong?

$\vec{F}(x,y) = \langle -y, x\rangle$

$\implies f_x = -y \implies f(x,y) = \int -y\,dx = -xy + C_1(y)$

$f_y = x \implies f(x,y) = \int x\,dy = xy + C_2(x)$

But there's no possible choice of $C_1(y)$, $C_2(x)$

since $f(x,y) = -xy + C_1(y)$ and $x = f_y = -x + C_1'(y)$

which means $C_1'(y) = 2x$, but this is impossible since C_1

can only be a function of y

Problem 4. Compute $\int_A \mathbf{F}\cdot d\mathbf{r}$ and $\int_C \mathbf{F}\cdot d\mathbf{r}$ for the vector field $\mathbf{F}(x,y) = \langle -y, x\rangle$ using the curves A and C from Problem 2. Is $\mathbf{F}$ path-independent?

(A) $\vec{r}(t) = \langle t, t\rangle$, $0 \le t \le 1$, $\vec{r}\,'(t) = \langle 1, 1\rangle$

(C) $\vec{r}(t) = \begin{cases} \langle t, 0\rangle & 0 \le t \le 1 \\ \langle 1, t-1\rangle, & 1 \le t \le 2 \end{cases}$ $\quad \vec{r}\,'(t) = \begin{cases} \langle 1, 0\rangle & 0 \le t \le 1 \\ \langle 0, 1\rangle & 1 \le t \le 2 \end{cases}$

$\int_A \vec{F}\cdot d\vec{r} = \int_0^1 \langle -t, t\rangle \cdot \langle 1, 1\rangle\,dt = 0$

$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \langle 0, t\rangle \cdot \langle 1, 0\rangle\,dt + \int_1^2 \langle 1-t, 1\rangle \cdot \langle 0, 1\rangle\,dt = 1$