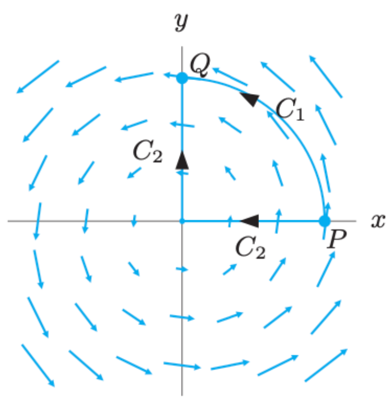
14.3, 14.4 Path-dependent Line Integrals and Green's Theorem

How to tell $\vec{F}$ is path-dependent

① using line integrals : find two paths C_1 and C_2 connecting A and B so that

$$\int_{C_1} \vec{F} \cdot d\vec{r} \neq \int_{C_2} \vec{F} \cdot d\vec{r}$$

Example



not path-indendent since $\int_{C_1} \vec{F} \cdot d\vec{r} \neq \int_{C_2} \vec{F} \cdot d\vec{r}$

Remark ① $\vec{F}$ is path-independent if and only if $\oint_C \vec{F} \cdot d\vec{r} = 0$ for all closed curves C.

② $\vec{F}$ is path-independent if and only if it's conservative.

② algebraically: check if it has a potential function

Example Does $\vec{F}(x,y) = \langle 2xy, xy \rangle$ have a potential function? If $f(x,y)$ is a potential function

$$f_x = 2xy \Rightarrow f(x,y) = \int f_x \, dx = \int 2xy \, dx = x^2 y + C(y).$$

Then

$$xy = f_y = x^2 + C'(y)$$

which implies $C'(y) = xy - x^2$, but $C'(y)$ should be a function of only y. No potential function, so path-dependent.

In general: $\vec{F} = \langle M, N \rangle$ is path-independent if and only if there exists f so that

$$M = f_x \quad \text{and} \quad N = f_y$$

If f_{xy} and f_{yx} are continuous,

$$M_y = f_{xy} = f_{yx} = N_x$$

Definition Let $\vec{F} = \langle M, N \rangle$ be a vector field in $\mathbb{R}^2$. The curl of $\vec{F}$ is the function $\operatorname{curl}(\vec{F}) = N_x - M_y$.

Theorem If $\operatorname{curl}(\vec{F}) \neq 0$, then $\vec{F}$ isn't conservative and is not path independent.

Warning If $\operatorname{curl}(\vec{F}) = 0$ we cannot yet conclude $\vec{F}$ is conservative (to be discussed).