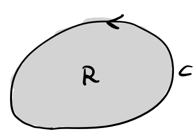
Definition Let $\vec{F} = \langle M, N \rangle$ be a vector field in $\mathbb{R}^2$. The **curl** of $\vec{F}$ is the function $\operatorname{curl}(\vec{F}) = N_x - M_y$.

There is a connection between $\oint_C F \cdot d\vec{r}$ and $\operatorname{curl}(\vec{F})$!

Theorem (Green's Theorem) Let C be a piecewise smooth, simple, closed curve that is the boundary of a region $R \subseteq \mathbb{R}^2$, positively oriented (which means counter-clockwise, so R is on the left as we move around C). Let $\vec{F} = \langle M, N \rangle$ be a smooth vector field with domain containing R and C.

Then
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \operatorname{curl}(\vec{F})\, dA$$

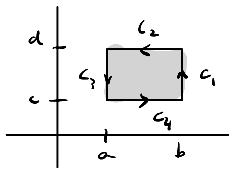


An example of C and R

Warning It's important that the domain of $\vec{F}$ includes R. For example if $\vec{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ and C is the positively oriented unit circle (so R is the unit disk), then we cannot apply Green's Theorem since $(0,0)$ is in R but not in the domain of $\vec{F}$.

Proof of Green's Theorem for rectangular region

Suppose R is the rectangle $a \le x \le b$, $c \le y \le d$ and C is the counter-clockwise boundary of R.



Let $\vec{F} = \langle M, N \rangle$.

Then $\iint_R \text{curl} F \, dA$

$\vec{r}_1(t) = \langle b, t \rangle, \; c \le t \le d$

$\vec{r}_2(t) = \langle t, d \rangle, \; a \le t \le b$

$\vec{r}_3(t) = \langle a, t \rangle, \; c \le t \le d$

$\vec{r}_4(t) = \langle t, c \rangle, \; a \le t \le b$

$$= \int_a^b \int_c^d (N_x - M_y) \, dy \, dx$$

$$= \int_c^d \int_a^b N_x \, dx \, dy - \int_a^b \int_c^d M_y \, dy \, dx$$

$$= \int_c^d (N(b,y) - N(a,y)) \, dy$$

$$- \int_a^b (M(x,d) - M(x,c)) \, dx.$$

And $\oint_C \vec{F} \cdot d\vec{r}$

$$= \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} + \int_{C_3} \vec{F} \cdot d\vec{r} + \int_{C_4} \vec{F} \cdot d\vec{r}$$

$$= \int_c^d \vec{F}(r_1(t)) \cdot r_1'(t) \, dt - \int_a^b \vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2'(t) \, dt$$

$$- \int_c^d \vec{F}(r_3(t)) \cdot r_3'(t) \, dt + \int_a^b \vec{F}(\vec{r}_4(t)) \, r_4'(t) \, dt$$

$$= \int_c^d N(b,t) \, dt - \int_a^b M(t,d) \, dt$$

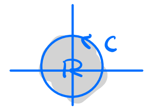
$$- \int_c^d N(a,t) \, dt + \int_a^b M(t,c) \, dt$$

$$= \int_c^d (N(b,t) - N(a,t)) \, dt - \int_a^b (M(t,d) - M(t,c)) \, dt$$

$$= \iint_R \text{curl} \, \vec{F} \, dA$$

Example Verify Green's Theorem for $\vec{F}(x,y) = \langle -y, x \rangle$

when C is the unit circle.



$\vec{r}(t) = \langle \cos t, \sin t \rangle, \quad 0 \le t \le 2\pi, \quad \vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt = \int_0^{2\pi} dt = 2\pi$$

$\operatorname{curl} \vec{F} = 1 - (-1) = 2$

$$\iint_R \operatorname{curl}(\vec{F})\, dA = \iint_R 2\, dA = 2\,\text{area}(R) = 2\pi$$

Problem 1. Verify Green's Theorem by computing $\oint_C \mathbf{F} \cdot d\mathbf{r}$ and $\iint_R \operatorname{curl} \mathbf{F}\, dA$ for each of the following vector fields $\mathbf{F}$ and closed, positively oriented curves C which enclose regions R.

a. $\mathbf{F}(x, y) = \langle x + y, y \rangle$ and C is the unit circle.

b. $\mathbf{F}(x, y) = \langle 3y, 2x \rangle$ and C is the parabola $x = y^2$ connecting $(1, 1)$ and $(1, -1)$ along with the line segment connecting these points.

(1) $\vec{r}(t) = \langle \cos t, \sin t \rangle, \quad 0 \le t \le 2\pi, \quad \vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \langle \cos t + \sin t, \sin t \rangle \cdot \langle -\sin t, \cos t \rangle$$

$$= -\sin t \cos t - \sin^2 t + \sin t \cos t = -\sin^2 t \, dt$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} -\sin^2 t \, dt = \int_0^{2\pi} \tfrac{1}{2}(\cos 2t - 1)\, dt$$

$$= \tfrac{1}{4} \sin 2t - \tfrac{1}{2} t \Big|_0^{2\pi}$$

$$= -\pi$$

(2) $\operatorname{curl} \vec{F} = -1$

$$\iint_R \operatorname{curl} F \, dA$$

$$= -\text{area}(R) = -\pi$$

(1) $\vec{r}_1(t) = \langle t^2, -t \rangle \quad -1 \le t \le 1, \quad \vec{r}_1{}'(t) = \langle 2t, -1 \rangle$

$\vec{r}_2(t) = \langle 1, t \rangle, \quad -1 \le t \le 1, \quad \vec{r}_2{}'(t) = \langle 0, 1 \rangle$

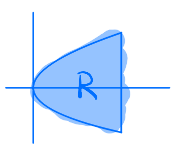
$$\vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t) = \langle -3t, 2t^2 \rangle \cdot \langle 2t, -1 \rangle = -6t^2 - 2t^2 = -8t^2$$

$$\vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2{}'(t) = \langle 3t, 2 \rangle \cdot \langle 0, 1 \rangle = 2$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{-1}^{1} (-8t^2 + 2)\, dt = 2\int_0^1 (-8t^2+2)\, dt = 2\left(-\frac{8}{3}t^2 + 2t \Big|_0^1\right)$$

$$= 2\left(-\frac{8}{3} + 2\right) = -\frac{4}{3}$$

(2)



$\text{curl}\, \vec{F} = 2 - 3 = -1$

$$\iint_R \text{curl}\, \vec{F}\, dA = -\int_{-1}^{1}\int_{y^2}^{1} dx\, dy = \int_{-1}^{1} (y^2 - 1)\, dy = 2\left(\frac{1}{3}y^3 - y \Big|_0^1\right)$$

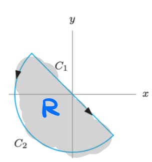
$$= 2\left(\frac{1}{3} - 1\right) = -\frac{4}{3}$$

Problem 2. Let $\mathbf{F}(x,y) = \langle y, x \rangle$ and $\mathbf{G}(x,y) = \langle 3y, -3x \rangle$. Let $C = C_2 - C_1$ as shown in the figure below, where C_1 is the line segment from $(-1,1)$ to $(1,-1)$ and C_2 is the portion of the positively oriented unit circle between these points.

a. Compute the curl of $\mathbf{F}$ and $\mathbf{G}$ and find the potential function of each vector field if possible.

b. State whether the Fundamental Theorem of Line Integrals or Green's Theorem can be used to compute each of the following line integrals. Compute the value of each using any valid method.

1. $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$
2. $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$
3. $\oint_C \mathbf{F} \cdot d\mathbf{r}$
4. $\int_{C_1} \mathbf{G} \cdot d\mathbf{r}$
5. $\int_{C_2} \mathbf{G} \cdot d\mathbf{r}$
6. $\oint_C \mathbf{G} \cdot d\mathbf{r}$



(a) $\text{curl}\, \vec{F} = 0, \quad \text{curl}\, \vec{G} = -6 \neq 0$

$\vec{G}$ does not have a potential function

for $\vec{F}$: $f_x = y \Rightarrow f(x,y) = xy + C(y)$

$f_y = x \Rightarrow x = x + C'(y)$

$\Rightarrow C'(y) = 0 \Rightarrow C'(y) = C$ for some C

So $f(x,y) = xy$ is a potential function.

(b)

integral	FTOLI	Green's Thm.	value
$\int_{C_1} \vec{F}\cdot d\vec{r}$	valid	not valid (C_1 not closed)	$f(1,-1)-f(-1,1)=0$
$\int_{C_2} \vec{F}\cdot d\vec{r}$	valid	not valid (C_2 not closed)	$f(1,-1)-f(-1,1)=0$
$\oint_{C} \vec{F}\cdot d\vec{r}$	valid	valid	0
$\int_{C_1} \vec{G}\cdot d\vec{r}$	not valid (G not conservative)	not valid (C_1 not closed)	See (A)
$\int_{C_2} \vec{G}\cdot d\vec{r}$	not valid (see above)	not technically valid but...	See (C)
$\oint_{C} \vec{G}\cdot dr$	not valid (see above)	Valid	See (B)

(A) $\vec{r}(t) = \langle t, -t \rangle, \quad -1 \le t \le 1$

$$\int_{C_1} \vec{G}\cdot d\vec{r} = \int_{-1}^{1} \langle -3t, -3t \rangle \cdot \langle 1, -1 \rangle \, dt$$

$$= \int_{-1}^{1} (-3t + 3t)\, dt = 0.$$

(B) $$\oint_C \vec{G}\cdot d\vec{r} = \iint_R \operatorname{curl}(\vec{G})\, dA = \iint_R -6\, dA = -6\,\text{area}(R) = -3\pi$$

(C) $$\oint_C \vec{G}\cdot d\vec{r} = \int_{C_2} \vec{G}\cdot d\vec{r} - \int_{C_1} \vec{G}\cdot d\vec{r} \quad \text{so}$$

$$\int_{C_2} \vec{G}\cdot d\vec{r} = \oint_C \vec{G}\cdot d\vec{r} + \int_{C_1} \vec{G}\cdot d\vec{r} = -3\pi$$