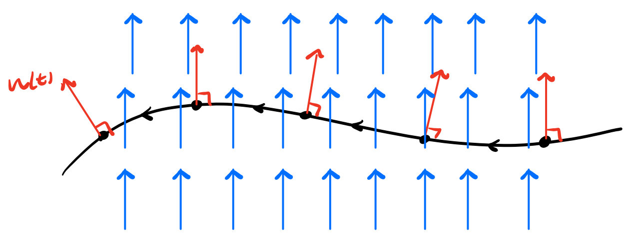
14.4 Flux in 2D

Warm-up remark if $\vec{v} = \langle a, b \rangle$, then $\vec{w} = \langle b, -a \rangle$ is the rotation of $\vec{v}$ by $\frac{\pi}{2}$ in the clockwise direction.

$\int_C \vec{F} \cdot d\vec{r}$ measures the the flow of $\vec{F}$ along C. Today we discuss a new line integral that measures the flow of $\vec{F}$ across C. We refer to this as the flux of F across C.



Let C be given by $\vec{r}(t) = \langle f(t), g(t) \rangle$, $a \le t \le b$.

Then $\vec{r}\,'(t) = \langle f'(t), g'(t) \rangle$ is tangent to C

and $\vec{n}(t) = \dfrac{\langle g'(t), -f'(t) \rangle}{\|\vec{r}\,'(t)\|}$ is perpendicular

and outward facing along C when C is closed and positively (counter-clockwise) oriented, and it's a unit vector.

Def The flux of $\vec{F}$ across C is given by

$$\boxed{\int_C \vec{F}\cdot\vec{n}\,ds = \int_a^b \vec{F}(\vec{r}(t))\cdot\langle g'(t), -f'(t)\rangle\,dt}$$

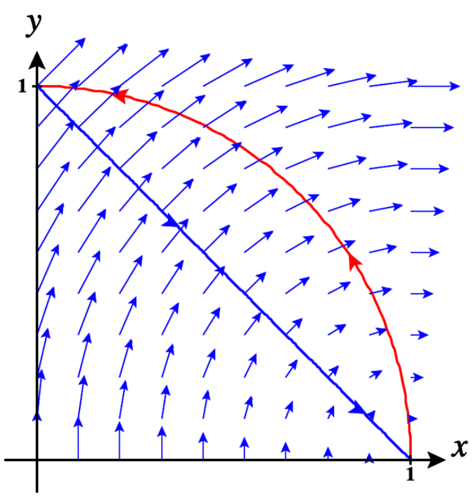
where $\vec{r}(t) = \langle f(t), g(t)\rangle$, $a \le t \le b$, is a parametrization of C.

Example Let $\vec{F}(x,y) = \langle y, 1-x\rangle$ and

C_1 the line segment from $(0,1)$ to $(1,0)$

C_2 the quarter unit circle from $(1,0)$ to $(0,1)$

Compute $\oint_C \vec{F}\cdot n\,ds = \int_{C_1}\vec{F}\cdot\vec{n}\,ds + \int_{C_2}\vec{F}\cdot\vec{n}\,ds.$



$\vec{r}_1(t) = \langle t, 1-t\rangle$, $0 \le t \le 1$

$\vec{r}_1'(t) = \langle 1, -1\rangle$

$\vec{r}_2(t) = \langle \cos t, \sin t\rangle$, $0 \le t \le \frac{\pi}{2}$

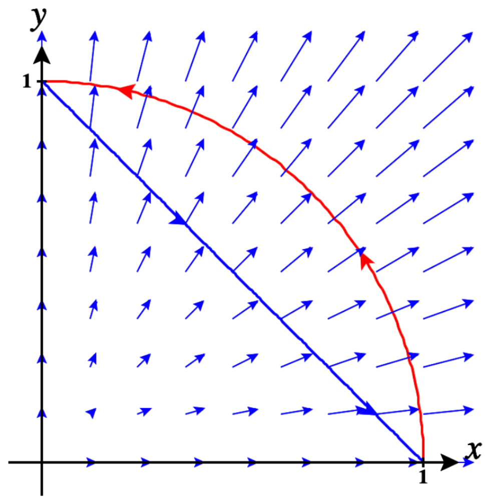
$\vec{r}_2'(t) = \langle -\sin t, \cos t\rangle$

$$\int_{C_1}\vec{F}\cdot\vec{n}\,ds = \int_0^1 \vec{F}(\vec{r}_1(t))\cdot\langle -1,-1\rangle\,dt$$
$$= \int_0^1 \langle 1-t, 1-t\rangle\cdot\langle -1,-1\rangle\,dt$$
$$= \int_0^1 2(t-1)\,dt$$
$$= t^2 - 2t\Big|_0^1 = -1$$

$$\int_{C_2}\vec{F}\cdot\vec{n}\,ds = \int_0^{\frac{\pi}{2}}\langle \sin t, 1-\cos t\rangle\cdot\langle \cos t, \sin t\rangle\,dt$$
$$= \int_0^{\pi/2}\sin t\,dt = -\cos t\Big|_0^{\pi/2} = 1$$

So $\oint_C \vec{F}\cdot\vec{n}\,ds = 0.$

Example Repeat the problem using $\vec{F}(x,y) = \langle x, y \rangle$.



$$\int_{C_1} \vec{F} \cdot \vec{n}\, ds = \int_0^1 \langle t, 1-t \rangle \cdot \langle -1, -1 \rangle\, dt$$

$$= \int_0^1 -1\, dt$$

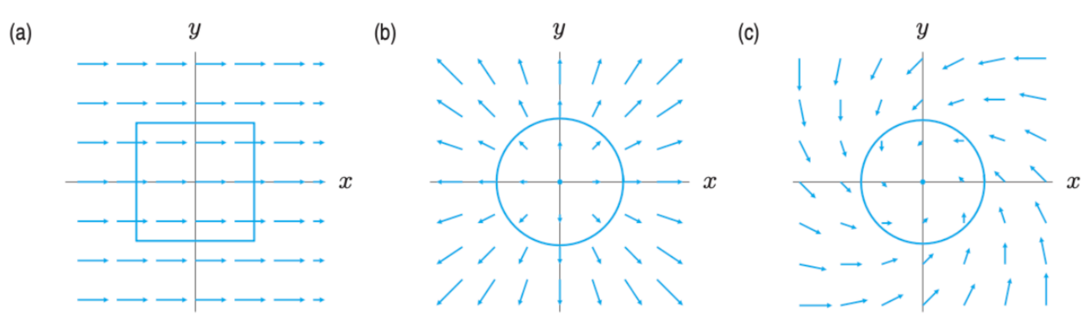
$$= -1$$

$$\int_{C_2} \vec{F} \cdot \vec{n}\, ds = \int_0^{\pi/2} \langle \cos t, \sin t \rangle \cdot \langle \cos t, \sin t \rangle\, dt$$

$$= \int_0^{\pi/2} 1\, dt = \frac{\pi}{2}.$$

So $\oint_C \vec{F} \cdot \vec{n}\, ds = \frac{\pi}{2} - 1$

Problem 1. Each image below shows a vector **F** along with a closed curve C that we should assume is positively oriented. Determine the sign (positive, negative, or zero) of $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$ in each case.



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Problem 2. Let $\mathbf{F}(x,y) = \langle x-y, x+y\rangle$. Let C_1 be the parabola $y = x^2$ from $(0,0)$ to $(2,4)$, let C_2 be the line segment from $(2,4)$ to $(0,0)$, and let $C = C_1 + C_2$. Sketch $\mathbf{F}$ in CalcPlot3d to estimate whether the flux integral $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds$ is positive, negative, or zero and then compute it.

$$\vec{r}_1(t) = \langle t, t^2\rangle, \quad 0 \le t \le 2$$

$$\vec{r}_2(t) = \langle t, 2t\rangle, \quad 0 \le t \le 2$$

$$\int_{C_1} \vec{F}\cdot\vec{n}\,ds = \int_0^2 \langle t-t^2, t+t^2\rangle \cdot \langle 2t, -1\rangle\,dt$$

$$= \int_0^2 (2t^2 - 2t^3 - t - t^2)\,dt$$

$$= \int_0^2 (t^2 - 2t^3 - t)\,dt$$

$$= \frac{1}{3}t^3 - \frac{1}{2}t^4 - \frac{1}{2}t^2 \Big|_0^2 = \frac{8}{3} - 8 - 2 = \frac{-22}{3}$$

$$\int_{C_2} \vec{F}\cdot\vec{n}\,ds = -\int_0^2 \langle -t, 3t\rangle \cdot \langle 2, -1\rangle\,dt$$

$$= \int_0^2 5t\,dt = \frac{5}{2}t^2 \Big|_0^2 = 10$$

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \frac{-22}{3} + 10 = \frac{8}{3}$$

Problem 3. Let $\mathbf{F}(x,y) = \langle -y, x\rangle$ and let C be the unit circle oriented counter-clockwise. Sketch $\mathbf{F}$ in CalcPlot3d to estimate whether the flux integral $\oint_C \mathbf{F}\cdot\mathbf{n}\,ds$ is positive, negative, or zero and then compute it.

$$\vec{r}(t) = \langle \cos t, \sin t\rangle, \quad 0 \le t \le 2\pi$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t\rangle$$

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \int_0^{2\pi} \langle -\sin t, \cos t\rangle \cdot \langle \cos t, \sin t\rangle\,dt$$

$$= 0$$