Curl Test (in 2D)

Let $\vec{F} = \langle M, N \rangle$ be a vector field with domain D.

① if $\vec{F}$ is conservative, then curl $\vec{F} = 0$.

(this means if curl $\vec{F} \neq 0$, then $\vec{F}$ is not conservative)

② if curl $\vec{F} = 0$ and D has no holes, then $\vec{F}$ is conservative.

③ if curl $\vec{F} = 0$ and D has holes, then $\vec{F}$ may or may not be conservative.

Proof of ① Suppose $\vec{F} = \langle M, N \rangle$ is conservative.

Then $\vec{F} = \nabla f$ for some function f. So

$$M = f_x \quad \text{and} \quad N = f_y.$$

Therefore since $f_{xy} = f_{yx}$, $M_y = N_x$. Thus

$$\text{curl}\,\vec{F} = N_x - M_y = 0.$$

Proof of ② Let C be piecewise smooth closed curve in D. Since D has no holes, we can apply Green's Theorem and say

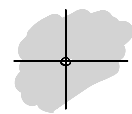
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl}\,\vec{F}\, dA = \iint_R 0\, dA = 0.$$

Therefore $\vec{F}$ is path-independent and so it's conservative.

Proof of (3)

We'll show that there exist vector fields $\vec{F}$ and $\vec{G}$ where $\operatorname{curl}\vec{F} = \operatorname{curl}\vec{G} = 0$ and each has a domain with a hole but one is conservative and one is not conservative.

Let $\vec{F}(x,y) = \dfrac{\langle -y, x \rangle}{\sqrt{x^2+y^2}}$ and $\vec{G} = \dfrac{\langle x, y \rangle}{\sqrt{x^2+y^2}}$. Then each has domam given by $\mathbb{R}^2 - \{(0,0)\}$.



Also, you can check that

(1) $\operatorname{curl}\vec{F} = \operatorname{curl}\vec{G} = 0$

(2) $\oint_C \vec{F} \cdot d\vec{r} = 2\pi$ when C is the unit circle

(which implies $\vec{F}$ is not path-independent and hence not conservative)

(3) $G = \nabla g$ where $g(x,y) = \frac{1}{2}\ln(x^2+y^2)$

(which implies G is conservative).