14.4 Divergence Theorem in 2D

Definition Let $\vec{F} = \langle M, N \rangle$ be a smooth vector field. Then the divergence of $\vec{F}$ is $\operatorname{div} \vec{F} = M_x + N_y$.

Theorem (Divergence Theorem) Let C be a piecewise smooth, positively oriented, closed curve that encloses a region $R \subseteq \mathbb{R}^2$. Let $\vec{F}$ be a smooth vector field whose domain includes R and C. Then

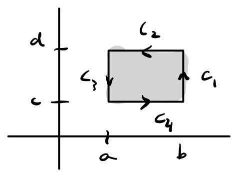
$$\oint_C F \cdot \vec{n}\, ds = \iint_R \operatorname{div} \vec{F}\, dA .$$

The macroscopic flux $\oint_C \vec{F} \cdot \vec{n}\, ds$ across C can be computed by summing microscopic fluxes throughout region R.

Proof of the Divergence Theorem for rectangles

Suppose R is the rectangle $a \le x \le b$, $c \le y \le d$

and C is the counter-clockwise boundary of R.



Let $\vec{F} = \langle M, N \rangle$.

Then $\displaystyle\iint_R \operatorname{div} \vec{F} \, dA$

$\vec{r}_1(t) = \langle b, t \rangle$, $c \le t \le d$

$\vec{r}_2(t) = \langle t, d \rangle$, $a \le t \le b$

$\vec{r}_3(t) = \langle a, t \rangle$, $c \le t \le d$

$\vec{r}_4(t) = \langle t, c \rangle$, $a \le t \le b$

$$= \int_a^b \int_c^d (M_x + N_y) \, dy \, dx$$

$$= \int_c^d \int_a^b M_x \, dx \, dy + \int_a^b \int_c^d N_y \, dy \, dx$$

$$= \int_c^d (M(b,y) - M(a,y)) \, dy$$

$$+ \int_a^b (N(x,d) - N(x,c)) \, dx.$$

And $\displaystyle\oint_C \vec{F} \cdot \vec{n} \, ds$

$$= \int_{C_1} \vec{F} \cdot \vec{n} \, ds + \int_{C_2} \vec{F} \cdot \vec{n} \, ds + \int_{C_3} \vec{F} \cdot \vec{n} \, ds + \int_{C_4} \vec{F} \cdot \vec{n} \, ds$$

$$= \int_c^d \vec{F}(r_1(t)) \cdot \langle 1, 0 \rangle \, dt - \int_a^b \vec{F}(\vec{r}_2(t)) \cdot \langle 0, -1 \rangle \, dt$$

$$- \int_c^d \vec{F}(r_3(t)) \cdot \langle 1, 0 \rangle \, dt + \int_a^b \vec{F}(\vec{r}_4(t)) \cdot \langle 0, -1 \rangle \, dt$$

$$= \int_c^d M(b,t) \, dt + \int_a^b N(t,d) \, dt$$

$$- \int_c^d M(a,t) \, dt - \int_a^b N(t,c) \, dt$$

$$= \int_c^d (M(b,t) - M(a,t)) \, dt - \int_a^b (N(t,d) - N(t,c)) \, dt$$

$$= \iint_R \operatorname{div} \vec{F} \, dA$$

Example Let $\vec{F}(x,y) = \langle -x, 2y-x \rangle$ and

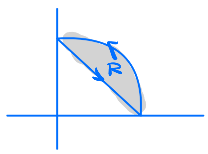
C_1 the quarter unit circle from $(1,0)$ to $(0,1)$,

C_2 the line segment from $(0,1)$ to $(1,0)$,

C the closed, positively oriented curve $C_1 + C_2$,

R the region enclosed by C. Verify the Divergence Theorem by computing both

$$\oint_C \vec{F}\cdot\vec{n}\,ds \quad \text{and} \quad \iint_R \operatorname{div}\vec{F}\,dA.$$



$\vec{r}_1(t) = \langle \cos t, \sin t \rangle,\ 0 \le t \le \frac{\pi}{2}$

$\vec{r}_1'(t) = \langle -\sin t, \cos t \rangle$

$\vec{r}_2(t) = \langle t, 1-t \rangle,\ 0 \le t \le 1$

$\vec{r}_2'(t) = \langle 1, -1 \rangle$

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \int_{C_1} \vec{F}\cdot\vec{n}\,ds + \int_{C_2} \vec{F}\cdot\vec{n}\,ds$$

$$= \int_0^{\frac{\pi}{2}} \langle -\cos t, 2\sin t - \cos t \rangle \cdot \langle \cos t, \sin t \rangle\,dt$$

$$+ \int_0^1 \langle -t, 2(1-t) - t \rangle \cdot \langle -1, -1 \rangle\,dt$$

$$= \int_0^{\frac{\pi}{2}} \left(-\cos^2 t + 2\sin^2 t - \sin t\cos t\right)dt$$

$$+ \int_0^1 \left(t + t - 2(1-t)\right)dt$$

$$= \frac{\pi}{4} - \frac{1}{2}.$$

$\operatorname{div}\vec{F} = -1 + 2 = 1$

$$\iint_R \operatorname{div}\vec{F}\,dA = \iint_R 1\,dA = \text{area}(R) = \frac{\pi}{4} - \frac{1}{2}$$

Problem 1. Let $\mathbf{F} = \langle x - y, x + y \rangle$, let C be the circle of radius 2 centered at the origin, and let R be the region enclosed by C. Verify the Divergence Theorem for this example. That is, compute $\oint_C \mathbf{F} \cdot \mathbf{n}\, ds$ using a parametrization and compute $\iint_R \operatorname{div} F\, dA$.

$$\vec{r}(t) = 2\langle \cos t, \sin t \rangle, \quad 0 \le t \le 2\pi, \quad \vec{r}\,'(t) = 2\langle -\sin t, \cos t \rangle$$

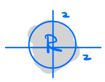
$$\vec{F}(\vec{r}(t)) \cdot \vec{n}(t)\, \|\vec{r}\,'(t)\| = 2\langle \cos t - \sin t, \cos t + \sin t \rangle \cdot 2\langle \cos t, \sin t \rangle$$

$$= 4(\cos^2 t - \sin t \cos t + \sin t \cos t + \sin^2 t)$$

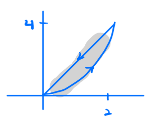
$$= 4$$

$$\oint_C \vec{F} \cdot \vec{n}\, ds = \int_0^{2\pi} 4\, dt = 8\pi$$

$$\operatorname{div} F = 1 + 1 = 2 \qquad \iint_R \operatorname{div} F\, dA = 2\, \text{area}(R) = 2\pi(2)^2 = 8\pi$$



Problem 2. Try Problem 1 using C as the closed, positively oriented curve comprising the parabola $y = x^2$ for $0 \le x \le 2$ and the line segment from $(2, 4)$ to $(0, 0)$.



$$\vec{r}_1(t) = \langle t, t^2 \rangle, \quad 0 \le t \le 2, \quad \vec{r}\,'(t) = \langle 1, 2t \rangle$$

$$\vec{r}_2(t) = \langle t, 2t \rangle, \quad 0 \le t \le 2, \quad \vec{r}\,'(t) = \langle 1, 2 \rangle$$

$$\vec{F}(\vec{r}_1(t)) \cdot \vec{n}_1(t)\, \|\vec{r}_1\,'(t)\| = \langle t - t^2, t + t^2 \rangle \cdot \langle 2t, -1 \rangle = 2t^2 - 2t^3 - t - t^2$$

$$= -2t^3 + t^2 - t$$

$$\vec{F}(\vec{r}_2(t)) \cdot \vec{n}_2(t)\, \|\vec{r}_2\,'(t)\| = \langle t - 2t, t + 2t \rangle \cdot \langle 2, -1 \rangle = -2t - 3t = -5t$$

$$\oint_C \vec{F} \cdot \vec{n}\, ds = \int_0^2 (-2t^3 + t^2 + 4t)\, dt = -\tfrac{1}{2}t^4 + \tfrac{1}{3}t^3 + 2t^2 \Big|_0^2$$

$$= -8 + \frac{8}{3} + 8 = \frac{8}{3}$$

$$\iint_R \operatorname{div} F\, dA = \int_0^2 \int_{x^2}^{2x} 2\, dy\, dx$$

$$= 2\int_0^2 (2x - x^2)\, dx = 2\left(x^2 - \tfrac{1}{3}x^3 \Big|_0^2 \right)$$

$$= 2\left(4 - \frac{8}{3}\right) = 2\left(\frac{12}{3} - \frac{8}{3}\right) = \frac{8}{3}$$

Problem 3. Let $\mathbf{F}$ be a vector field whose domain is all of $\mathbb{R}^2$ with the property that $\operatorname{div} \mathbf{F} = 0$ and let C_1 and C_2 be two non-intersecting curves each oriented so that they start from $(0, 0)$ and go to $(1, 1)$. Suppose $\int_{C_1} \mathbf{F} \cdot \mathbf{n}\, ds = 5$. Find the value of $\int_{C_2} \mathbf{F} \cdot \mathbf{n}\, ds$ and explain your reasoning.

Note if R is the region enclosed by C_1 and C_2, $\iint_R \operatorname{div} F\, dA = 0$ since $\operatorname{div} F = 0$.

The divergence theorem tells us $0 = \iint_R \operatorname{div} \vec{F}\, dA = \oint_C \vec{F} \cdot \vec{n}\, ds = \int_{C_1} \vec{F} \cdot \vec{n}\, ds - \int_{C_2} \vec{F} \cdot \vec{n}\, ds$

$$= 5 - \int_{C_2} \vec{F} \cdot \vec{n}\, ds$$

Therefore $\int_{C_2} \vec{F} \cdot \vec{n}\, ds = 5$.