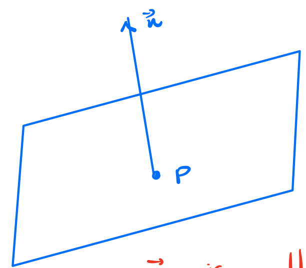
10.6 Planes

Goal Given a point $P = (x_0, y_0, z_0)$ and a vector $\vec{n} = \langle a, b, c \rangle$ write an equation for the plane passing through P and orthogonal to $\vec{n}$



$\vec{n}$ is called a normal vector for the plane.

Standard form equation The plane consists of all points $Q = (x, y, z)$ so that $\vec{PQ}$ and $\vec{n}$ are orthogonal:

$$0 = \vec{PQ} \cdot \vec{n}$$

$$= \langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle$$

$$\Rightarrow \quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

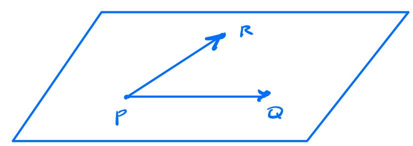
General form equation of plane

$$ax + by + cz = d$$

where $d = ax_0 + by_0 + cz_0$.

Example Let $P = (3,-1,2)$, $Q = (8,2,4)$, $R = (-1,-2,-3)$.

Find standard form and general form equations of the plane containing P, Q, R.



$\vec{v} = \overrightarrow{PQ} = \langle 5, 3, 2 \rangle$

$\vec{w} = \overrightarrow{PR} = \langle -4, -1, -5 \rangle$

$$\vec{n} = \vec{v} \times \vec{w}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & 3 & 2 \\ -4 & -1 & -5 \end{vmatrix}$$

$$= \langle -13, 17, -17 \rangle$$

$-13(x-3) + 17(y+1) - 17(z-2) = 0$ std form

$-13x + 17y - 17z = -90$ general form

Example Consider the plane $2x + y + 3z = 1$.

① Find a normal vector $\vec{n}$.

② Find a point P on plane

③ Find a vector orthogonal to $\vec{n}$.

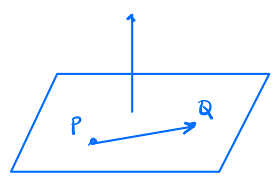
① $\vec{n} = \langle 2, 1, 3 \rangle$

② If $x = 1$, $y = 2$,

$2(1) + (2) + 3z = 1 \Rightarrow z = -1$

$\vec{P} = (1, 2, -1)$

③



any vector $\vec{PQ}$ on plane is orthogonal to $\vec{n}$, so let's find another point Q on the plane : $x = 0$, $y = -2 \Rightarrow 2(0) + (-2) + 3z = 1$

$Q = (0, -2, 1)$

$\vec{PQ} = \langle -1, -4, 2 \rangle$ works (check that $\vec{PQ} \cdot \vec{n} = 0$)

Problem 1. Give a standard form equation of the plane containing the points $(1, 2, 3)$, $(3, -1, 4)$ and $(1, 0, 1)$.

$P = (1, 2, 3)$, $Q = (3, -1, 4)$, $R = (1, 0, 1)$

$\vec{PQ} = \langle 2, -3, 1 \rangle$, $\vec{PR} = \langle 0, -2, -2 \rangle$

$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 1 \\ 0 & -2 & -2 \end{vmatrix} = \langle 8, 4, -4 \rangle$$

$$8(x-1) + 4(y-2) - 4(z-3) = 0$$

Problem 2. Give a standard form equation of the plane that contains the intersecting lines $\ell_1(t) = \langle 2, 1, 2 \rangle + t \langle 1, 2, 3 \rangle$ and $\ell_2(t) = \langle 2, 1, 2 \rangle + t \langle 2, 5, 4 \rangle$.

$\vec{v} = \langle 1, 2, 3 \rangle$, $\vec{w} = \langle 2, 5, 4 \rangle$

$$\vec{n} = \vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 2 & 5 & 4 \end{vmatrix} = \langle -7, 2, 1 \rangle$$

$$-7(x-2) + 2(y-1) + (z-2) = 0$$

Problem 3. Give a standard form equation of the plane containing the point $(1, 2, 3)$ that is parallel to the plane $2x + 4y + 6z = 8$.

$\vec{n} = \langle 2, 4, 6 \rangle$

$$2(x-1) + 4(y-2) + 6(z-3) = 0$$

Problem 4. Consider the line $\ell(t) = \langle 5, 1, -1\rangle + t\,\langle 2, 2, 1\rangle$ and the plane $5x - y - z = -3$.

a. Compute the dot product between $\mathbf{v} = \langle 2, 2, 1\rangle$ and a normal vector $\mathbf{n}$ for the plane.

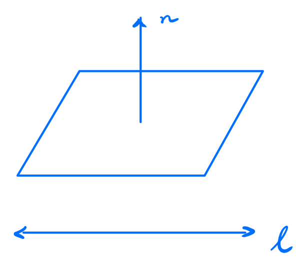
b. Use your previous answer to explain why the line and plane are not parallel.

c. Since the line and plane are not parallel, they must intersect. Find the point of intersection by finding a value t so that $\ell(t)$ is a point on the plane.

(a) $\vec{n} = \langle 5, -1, -1\rangle$

$\vec{n} \cdot \vec{v} = 10 - 2 - 1 = 7 \neq 0$

(b) If line and plane parallel, $\vec{v}$ and $\vec{n}$ are orthogonal. But $\vec{v} \cdot \vec{n} \neq 0$ means they're not orthogonal.



(c) Find t so that

$(5+2t,\ 1+2t,\ -1+t)$ satisfies

$5x - y - z = -3$:

$5(5+2t) - (1+2t) - (-1+t) = -3$

$\Rightarrow \quad 25 + 10t - 1 - 2t + 1 - t = -3$

$\Rightarrow \quad 7t = -28$

$t = -4$ So $(-3, -7, -5)$ is point of intersection.