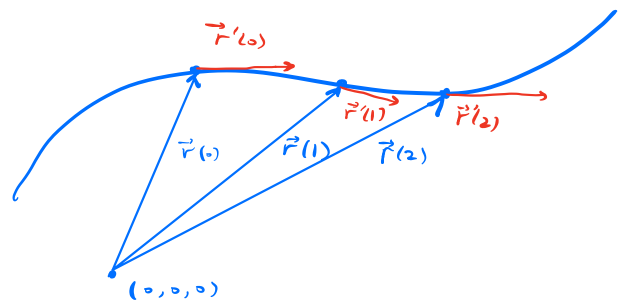
11.2 Calculus and Vector-Valued Functions

Given a vector-valued function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$,

it's derivative is also a vector-valued function:

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

(as long as f, g, h are differentiable).



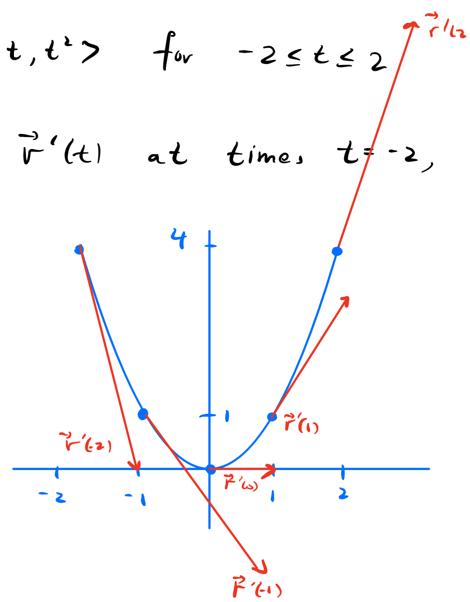
$\vec{r}\,'(t)$ is a tangent vector to the curve at the point $\vec{r}(t)$ on the curve

Example Let $\vec{r}(t) = \langle t, t^2 \rangle$ for $-2 \le t \le 2$

Plot $\vec{r}(t)$ and plot $\vec{r}\,'(t)$ at times $t = -2$, $-1, 0, 1$, and 2.

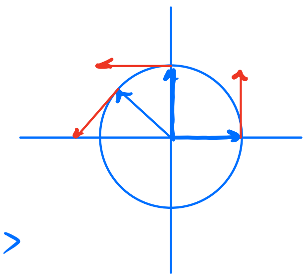
$\vec{r}\,'(t) = \langle 1, 2t \rangle$

t	$\vec{r}\,'(t)$
-2	$\langle 1, -4 \rangle$
-1	$\langle 1, -2 \rangle$
0	$\langle 1, 0 \rangle$
1	$\langle 1, 2 \rangle$
2	$\langle 1, 4 \rangle$



Example Let $\vec{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \le t \le 2\pi$.

Compute $\vec{r}(t) \cdot \vec{r}\,'(t)$.



$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

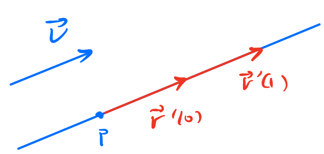
$\vec{r}(t) \cdot \vec{r}\,'(t) = \langle \cos t, \sin t \rangle \cdot \langle -\sin t, \cos t \rangle$

$= -\sin t \cos t + \sin t \cos t = 0$

Tangent vectors always orthogonal to $\vec{r}(t)$ vectors.

Example Let $\vec{r}(t) = \langle x_0 + tv_1, y_0 + tv_2, z + tv_3 \rangle$ be the line passing through $P = (x_0, y_0, z_0)$ in the direction of $\vec{v} = \langle v_1, v_2, v_3 \rangle$. Compute $\vec{r}\,'(t)$.

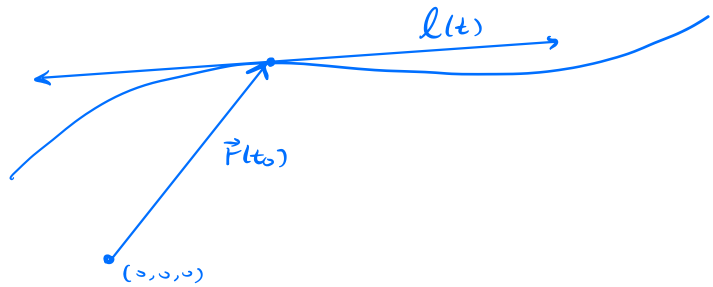
$\vec{r}\,'(t) = \langle v_1, v_2, v_3 \rangle = \vec{v}$.



$\vec{r}\,'(t)$ is constant $\vec{v}$.

same tangent vector at every point along line

Tangent lines



Given curve $\vec{r}(t)$, we want to find a vector equation $\ell(t)$ for the tangent line to the curve at time $t=c$

point on line : $\vec{p} = \vec{r}(c)$

direction of line : $\vec{v} = \vec{r}\,'(c)$

$$\ell(t) = \vec{r}(c) + t\,\vec{r}\,'(c).$$

Example Find tangent line to $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ at time $t=-1$.

$\vec{r}(-1) = \langle -1, 1, -1 \rangle$,

$\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$,

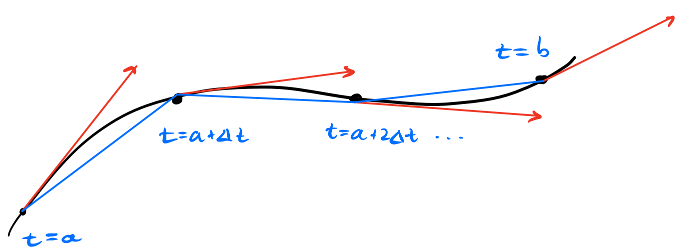
$\vec{r}\,'(-1) = \langle 1, -2, 3 \rangle$

$\ell(t) = \langle -1, 1, -1 \rangle + t\langle 1, -2, 3 \rangle$

Remark $\vec{r}\,'(t)$ represents velocity of particle whose position is given by $\vec{r}(t)$ at time t

$\|\vec{r}\,'(t)\|$ represents its speed at time t.

Arc length



The length of $\vec{r}(t)$ from $t=a$ to $t=b$

$\approx$ sum of lengths of piecewise line segments

$\approx$ sum of lengths $\|\vec{r}\,'(t)\|\,\Delta t$

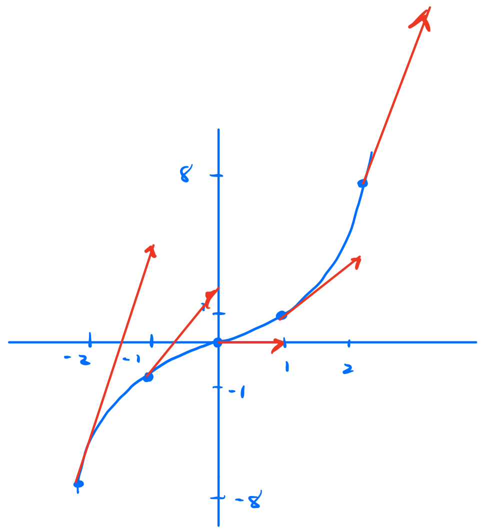
These approximate $\approx$ signs become more accurate line segments get shorter $(\Delta t \to 0)$

Formula Length of $\vec{r}(t)$ from $t=a$ to $t=b$ is given by $\displaystyle\int_a^b \|\vec{r}\,'(t)\|\,dt$.

Problem 1. Plot the curve given by $\mathbf{r}(t) = \langle t, t^3 \rangle$ for $-2 \le t \le 2$ and plot the tangent vectors $\mathbf{r}'(t)$ at times $t = -2, -1, 0, 1, 2$.

$\vec{r}\,'(t) = \langle 1, 3t^2 \rangle$

t	$\vec{r}(t)$	$\vec{r}\,'(t)$
-2	$\langle -2, -8 \rangle$	$\langle 1, 12 \rangle$
-1	$\langle -1, -1 \rangle$	$\langle 1, 3 \rangle$
0	$\langle 0, 0 \rangle$	$\langle 1, 0 \rangle$
1	$\langle 1, 1 \rangle$	$\langle 1, 3 \rangle$
2	$\langle 2, 8 \rangle$	$\langle 1, 12 \rangle$

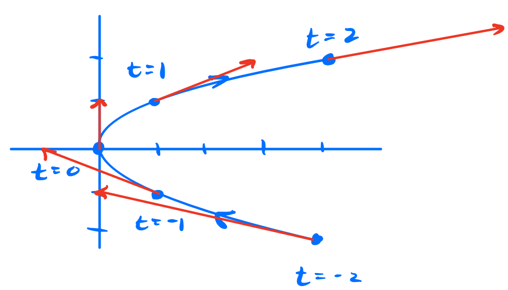


$x = t,\ y = t^3 \implies y = x^3$

Problem 2. Plot the curve given by $\mathbf{r}(t) = \langle t^2, t \rangle$ for $-2 \le t \le 2$ and plot the tangent vectors $\mathbf{r}'(t)$ at times $t = -2, -1, 0, 1, 2$.

$\vec{r}\,'(t) = \langle 2t, 1 \rangle$

t	$\vec{r}(t)$	$\vec{r}\,'(t)$
-2	$\langle 4, -2 \rangle$	$\langle -4, 1 \rangle$
-1	$\langle 1, -1 \rangle$	$\langle -2, 1 \rangle$
0	$\langle 0, 0 \rangle$	$\langle 0, 1 \rangle$
1	$\langle 1, 1 \rangle$	$\langle 2, 1 \rangle$
2	$\langle 4, 2 \rangle$	$\langle 4, 1 \rangle$



$x = t^2,\ y = t \implies x = y^2$

Problem 3. Find a vector equation for the tangent line to curve $\mathbf{r}(t)$ at the given t value.

a. $\mathbf{r}(t) = \langle 3t^3 - 2t^2 + t + 1, t^4 + t^3 - 3t \rangle, t = 1$

b. $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, \frac{t}{2\pi} \rangle, t = \pi$

(a) $\vec{r}\,'(t) = \langle 9t^2 - 4t + 1, 4t^3 + 3t^2 - 3 \rangle$

$\vec{r}(1) = \langle 3, -1 \rangle \quad \vec{r}\,'(1) = \langle 6, 4 \rangle$

$\ell(t) = \langle 3, -1 \rangle + t \langle 6, 4 \rangle$

(b) $\vec{r}\,'(t) = \langle -3\sin t, 3\cos t, \frac{1}{2\pi} \rangle$

$\vec{r}(\pi) = \langle -3, 0, \frac{1}{2} \rangle, \quad \vec{r}\,'(\pi) = \langle 0, -3, \frac{1}{2} \rangle$

$\ell(t) = \langle -3, 0, \frac{1}{2} \rangle + t \langle 0, -3, \frac{1}{2} \rangle.$

Problem 4. Compute $\mathbf{r}'(t)$ for the following functions.

a. $\mathbf{r}(t) = \langle \cos t, e^t, \ln t \rangle$

b. $\mathbf{r}(t) = \left\langle \sin(2t), e^{t^2}, \cos(t^3) \right\rangle$ (remember the chain rule)

c. $\mathbf{r}(t) = \langle t\cos t, t^2 \sin t, t^3 \ln t \rangle$ (remember the product rule)

(a) $\vec{r}\,'(t) = \langle -\sin t, e^t, \frac{1}{t} \rangle$

(b) $\vec{r}\,'(t) = \langle 2\cos(2t), 2te^{t^2}, -3t^2 \sin(t^3) \rangle$

(c) $\vec{r}\,'(t) = \langle \cos t - t\sin t, 2t\sin t + t^2 \cos t, 3t^2 \ln t + t^2 \rangle$