Closed sets

Def Let $A \subseteq \mathbb{R}$. Then A is a closed set if A^c is open.

Example $[0,1]$ is a closed set.

Proof To show $[0,1]$ is closed, we must show that $[0,1]^c$ is open. Observe that $[0,1]^c = (-\infty, 0) \cup (1, \infty)$ is the union of two open sets. In the previous lecture we proved that the union of open sets is always open.

Example $\mathbb{Z}$ is a closed set.

Proof Observe that $\mathbb{Z}^c = \bigcup_{n=-\infty}^{\infty} (n, n+1)$.

Therefore $\mathbb{Z}^c$ is the union of open sets and hence is open. Therefore $\mathbb{Z}$ is closed

Problem 1. Make a conjecture about which of the following are closed sets. Right now, instead of writing a full formal proof, can you outline an argument explaining your conjecture?

a. $\emptyset$

b. $\mathbb{R}$

c. $\{1\}$

d. $[0, 1)$

e. $(-\infty, 0]$

f. $\bigcap_{n=1}^{10}[1 - 1/n, 1 + 1/n]$

g. $\bigcap_{n=1}^{\infty}[1 - 1/n, 1 + 1/n]$

all except (d) are closed sets

Problem 2. Prove or disprove: the intersection of an infinite collection of closed sets is closed.

Let $\{A_\alpha : \alpha \in I\}$ be a collection of closed sets.

We claim $\bigcap_{\alpha \in I} A_\alpha$ is closed. Observe that

$\left(\bigcap_{\alpha \in I} A_\alpha\right)^c = \bigcup_{\alpha \in I} A_\alpha^c$. Moreover A_α^c is open for all $\alpha \in I$. Since the union of an arbitrary collection of open sets is open, $\bigcup_{\alpha \in I} A_\alpha^c$ is open.

Problem 3. Prove or disprove: the union of an infinite collection of closed sets is closed.

Counterexample: Let $A_n = (-\infty, 1-\frac{1}{n}] \cup [1+\frac{1}{n}, \infty)$ for each $n \in \mathbb{Z}^+$. Then A_n is closed for all $n \in \mathbb{Z}^+$.

However $\bigcup_{n=1}^{\infty} A_n$ is not closed since $\left(\bigcup_{n=1}^{\infty} A_n\right)^c$

$= \bigcap_{n=1}^{\infty} A_n^c = \bigcap_{n=1}^{\infty} (1-\frac{1}{n}, 1+\frac{1}{n}) = \{1\}$ is not open.

Problem 4. Write a summary of what you know about finite and infinite collections of open sets. Are intersections of open sets always open? Are unions of open sets always open?

Problem 5. Repeat the previous problem with closed sets.

Finite intersections/unions of open sets are open.

Finite intersection/unions of closed sets are closed.

Infinite unions of open sets are open.

Infinite unions of closed sets are not necessarily closed.

Infinite intersections of open sets are not necessarily open.

Infinite intersections of closed sets are closed.