# Chapter 9 The Power Set and the Cartesian Product

Def Let $S$ be a given set. Then the power set $\mathcal{P}(S)$ of $S$ is a collection of sets such that $A \in \mathcal{P}(S)$ if and only if $A \subseteq S$.

Example Let $A = \{0,1\}$. Then

$$\mathcal{P}(A) = \{\phi, \{0\}, \{1\}, A\}.$$

Let $B = \{1,2,3\}$. Then

$$\mathcal{P}(B) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, B\}$$

Definition Let $X$ and $Y$ be given sets. Their Cartesian product is the set ordered pairs

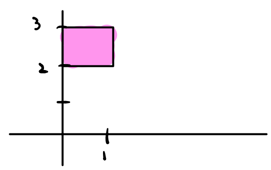
$$X \times Y = \{(x,y) : x \in X, y \in Y\}$$

where by ordered pair we mean a 2-element ordered list.

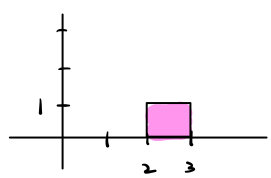
**Examples** ① $\{0,1\} \times \{2,3\} = \{(0,2),(0,3),(1,2),(1,3)\}$

② $\{2,3\} \times \{0,1\} = \{(2,0),(2,1),(3,0),(3,1)\}$

③ $[0,1] \times [2,3] = \{(x,y) : 0 \leq x \leq 1,\ 1 \leq y \leq 3\}$




④ $[2,3] \times [0,1] = \{(x,y) : 2 \leq x \leq 3,\ 0 \leq y \leq 1\}$




**Theorem** Let $A, B, C, D$ be sets. Then

$$(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D).$$

**Proof** Let $z \in (A \times B) \cup (C \times D)$. Then $z = (a,b)$ where $a \in A, b \in B$ or $z = (c,d)$ where $c \in C, d \in D$.

Consider the first case. Since $a \in A$, $a \in A \cup B$ and since $b \in B$, $b \in B \cup D$. Therefore $z = (a,b)$ where $a \in A \cup B$ and $b \in (B \cup D)$ and so $z \in (A \cup C) \times (B \cup D)$.

A similar argument holds for the second case.

Thus the desired inclusion holds.

**Problem 1.** Let $A$ be a set and $\mathcal{P}(A)$ its power set. Determine which of the following statements are true.

a. $A \in \mathcal{P}(A)$

b. $\emptyset \subseteq \mathcal{P}(A)$

c. $\emptyset = \mathcal{P}(\emptyset)$

d. $\{\emptyset\} = \mathcal{P}(\emptyset)$

e. If $a \in A$ then $\{a\} \subseteq \mathcal{P}(A)$

(a), (b), (d) are the only true statements

The following edits could be made to make true statements:

(c) $\phi \in \mathcal{P}(\phi)$

(e) if $a \in A$, then $\{a\} \in \mathcal{P}(A)$. or if $a \in A$, then $\{\{a\}\} \subseteq \mathcal{P}(A)$.

**Problem 2.** Consider the following expressions. In each, replace □ by either the symbol $\in$ or the symbol $\subseteq$ to make a correct statement.

a. $\mathbb{N} \square \mathbb{Z}$

b. $\{1, 2, 3\} \square \mathbb{Z}$

c. $\{5\} \square \mathbb{Z}$

d. $\mathbb{Z}^+ \square \mathcal{P}(\mathbb{Z})$

e. $\{1, 2, 3\} \square \mathcal{P}(\{1, 2, 3\})$

f. $5 \square \mathbb{Z}$

g. $\emptyset \square \mathbb{Z}$

h. $\emptyset \square \mathcal{P}(\mathbb{Z})$

$\subseteq$ : (a), (b), (c), (g), (h)

$\in$ : (d), (e), (f), (h)

**Problem 3.** Give an informal geometric description of the following sets. Are they same set?

a. $\mathbb{Z} \times \mathbb{R}$

b. $\mathbb{R} \times \mathbb{Z}$

not the same set:

(a) vertical lines in xy-plane of the form $x = n$, $n \in \mathbb{Z}$.

(b) horizontal lines in xy-plane of the form $y = n$, $n \in \mathbb{Z}$.

**Problem 4.** Let $A$ be any set. Make a conjecture about how to simplify $A \times \emptyset$. Explain your simplification.

$A \times \phi = \phi$ (see Theorem 9.6)

**Problem 5.** Let $A, B, C, D$ be sets. The following statement is false

$$(A \cup C) \times (B \cup D) \subseteq (A \times B) \cup (C \times D).$$

Use your conjecture from Problem 4 to find a counterexample.

$A = \{1\}$, $B = \phi$, $C = \phi$, $D = \{2\}$

Then $(A \cup C) \times (B \cup D) = \{(1, 2)\}$ but $(A \times B) \cup (C \times D) = \phi$.