Problem 1. Negate the following statements:

a. For all $x > 0$ there exists $y > 0$ such that $xy = 1$.

b. For every prime number p there is another prime number q such that $q > p$.

c. We have $0 < x < 10$ and it is also the case that $x \leq 2$ or $x \geq 3$.

(a) There exists $x > 0$ such that for all $y > 0$, $xy \neq 1$.

(b) There exists a prime number p such that for all primo numbers q, $q \leq p$.

(c) It is the case that $x \in (-\infty, 0] \cup [10, \infty)$ or $x \in (2,3)$. In other words, $x \in (-\infty, 0] \cup (2,3) \cup [10, \infty)$.

Problem 2. Write the converse and the contrapositive of the following statements:

a. If $x > -3$ then $|(x+3)/(x+5)| < 1$.

b. If $x \in A \cup B$ then $x \in A$ or $x \in B$.

(a) Converse: If $\left|\frac{x+3}{x+5}\right| < 1$, then $x > -3$.

Contrapositive: If $\left|\frac{x+3}{x+5}\right| \geq 1$, then $x \leq -3$.

(b) Converse: If $x \in A$ or $x \in B$, then $x \in A \cup B$.

Contrapositive: If $x \notin A$ and $x \notin B$, then $x \notin A \cup B$.

Problem 3. State whether each of the following is true or false.

a. For all $x \in \mathbb{Q}$ there exists $y \in \mathbb{Q}$ such that $x = 2y$.

b. There exists $y \in \mathbb{Q}$ such that for all $x \in \mathbb{Q}$, $x = 2y$.

c. For all $x \in \mathbb{Q}$ there exists $y \in \mathbb{Q}$ such that $x = y^2$.

d. The set $\bigcap_{n=1}^{\infty}(-1/n, 1 + 1/n)$ is open.

e. The set $\bigcup_{n=1}^{\infty}(-1/n, 1 + 1/n)$ is open.

(a) True: given $x \in \mathbb{Q}$, let $y = \frac{x}{2}$ and notice $\frac{x}{2} \in \mathbb{Q}$.

(b) False: not possible to find a y that works for all x.

(c) False: counterexample is $x = 2$.

(d) False: $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1+\frac{1}{n}\right) = [0,1]$, which is not open since when $x=0$ or $x=1$, for all $\delta > 0$, $(x-\delta, x+\delta) \not\subseteq [0,1]$.

(e) True: $\bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, 1+\frac{1}{n}\right) = (-1, 2)$ is open since any interval of the form (a,b) where $a, b \in \mathbb{R}$ was proven to be open in homework.

Problem 4. Prove the following statements:

a. Let $a, b, c \in \mathbb{Z}$. If $a^2 \mid b$ and $b^3 \mid c$, then $a^6 \mid c$.

b. Let $a, b \in \mathbb{Z}$. If a and b have the same parity (ie. they're both even or both odd), then $3a + 7$ and $7b - 4$ do not have the same parity.

c. Let $\delta \in (0, 1)$ and suppose that $|x - 3| < \delta$. Prove that there exists a constant C, which does not depend on x nor on δ, such that $|2x^2 - 18| \leq C\delta$.

d. Let A, B, C be sets. If $A \subseteq B$ then $A \setminus C \subseteq B \setminus C$.

(a) Since $a^2 \mid b$ and $b^3 \mid c$, $b = a^2 n$ and $c = b^3 m$ for some $n, m \in \mathbb{Z}$. Therefore $c = (a^2 n)^3 m = a^6 n^3 m$. Since $n^3 m \in \mathbb{Z}$, we have $a^6 \mid c$.

(b) We will do a proof by two cases. In the first case we assume a and b are both even. Thus $a = 2n$ and $b = 2m$ for some $n, m \in \mathbb{Z}$.

Observe that $3a+7 = 6n+7 = 2(3n+3)+1$, which shows $3a+7$ is odd. Moreover, $7b-4 = 14m-4 = 2(7m-2)$ which shows $7b-4$ is even. Thus $3a+7$ and $7b-4$ have opposite parity in this case.

In the second case we assume a and b are both odd. That is $a = 2n+1$ and $b = 2m+1$ for some $n, m \in \mathbb{Z}$. Then $3a+7 = 3(2n+1)+7 = 6n+10 = 2(3n+5)$ and $7b-4 = 7(2m+1)-4 = 14m+3 = 2(7m+1)+1$

which shows they are even and odd respectively. Therefore they again have opposite parity.

(c) Let $C = 14$. Observe that

$$\begin{aligned}
|2x^2 - 18| &= 2|x^2 - 9| \\
&= 2|x-3||x+3| \\
&< 2\delta|x+3| \\
&\le 2\delta(|x| + 3) \\
&= 2\delta(|x-3+3| + 3) \\
&\le 2\delta(|x-3| + 6) \\
&< 2\delta(\delta + 6) \\
&< 2\delta(1+6) \\
&= C\delta.
\end{aligned}$$

(d) Let $x \in A \setminus C$. Then since $x \in A$ and $A \subseteq B$, $x \in B$. Moreover, notice $x \notin C$. Therefore, since $x \in B$ and $x \notin C$, $x \in B \setminus C$. Thus $A \setminus C \subseteq B \setminus C$.