Chapter 11 Partitions

Def A <u>partition</u> of a nonempty set X is a collection $\mathcal{A}$ of subsets of X satisfying:

① Every set $A \in \mathcal{A}$ is nonempty

② $\bigcup_{A \in \mathcal{A}} A = X$

③ for all $A, B \in \mathcal{A}$, if $A \cap B \neq \emptyset$, then $A = B$.

(this says partition elements are disjoint sets)

Examples ① $\mathcal{A} = \{ \mathbb{Z}^-, \{0\}, \mathbb{Z}^+ \}$ is a partition of $\mathbb{Z}$

② $\mathcal{B} = \{ [n, n+1) : n \in \mathbb{Z} \}$ is a partition of $\mathbb{R}$

③ $\mathcal{C} = \{ [-n, n] : n \in \mathbb{Z} \}$ is <u>not</u> a partition of $\mathbb{R}$

(fails condition ③)

Theorem 1 Let $\sim$ be an equivalence relation on a nonempty set X. Then the collection

$$\mathcal{A} = \{ E_x : x \in X \}$$

of equivalence classes of $\sim$ is a partition of X.

"Every equivalence relation gives rise to a partition."

<u>Lemma</u> Let $\sim$ be an equivalence relation on a nonempty set X. If $x, y \in X$ and $E_x \cap E_y \neq \emptyset$ then $E_x = E_y$.

<u>Proof of Lemma</u> We first show $E_x \subseteq E_y$. Let $z \in E_x$. Thus $z \sim x$. Since $E_x \cap E_y \neq \emptyset$, there exists $w \in X$ such that $w \sim x$ and $w \sim y$. Since $z \sim x$ and $x \sim w$ and $w \sim y$, using the transitivity of $\sim$ we have $z \sim y$. Thus $z \in E_y$. The same argument shows $E_y \subseteq E_x$.

<u>Proof of Theorem 1</u> We must show $\{E_x : x \in X\}$ satisfies the three conditions of a partition. Notice that for each $x \in X$, by the reflexive property of $\sim$, $x \sim x$, which implies $x \in E_x$. Thus E_x is nonempty for each $x \in X$. Next we claim $\bigcup_{x \in X} E_x = X$. It is clear that $\bigcup_{x \in X} E_x \subseteq X$ since $E_x \subseteq X$ for all $x \in X$. On the other hand, if $x \in X$, then $x \in E_x \subseteq \bigcup_{x \in X} E_x$ which shows $X \subseteq \bigcup_{x \in X} E_x$. Finally, property ③ of the definition is simply the statement of our lemma.

Theorem 2 Let $\mathcal{A}$ be a partition of a nonempty set X. Define a relation on X by $x \sim y$ if and only if $x, y \in A$ for some $A \in \mathcal{A}$ (ie. they're in the same partition element). Then $\sim$ is an equivalence relation.

"Every partition gives rise to an equivalence relation."

Proof of Theorem 2 We must show that $\sim$ is reflexive, symmetric, and transitive. For reflexivity, suppose $x \in X$. Since $X = \bigcup_{A \in \mathcal{A}} A$, $x \in A$ for some $A \in \mathcal{A}$. Since x is in the same partition set A as itself, $x \sim x$.

For symmetry, suppose $x, y \in X$ such that $x \sim y$. Then $x, y \in A$ for some $A \in \mathcal{A}$. Thus $y, x \in A$ and so $y \sim x$.

For transitivity, suppose $x, y, z \in X$ such that $x \sim y$ and $y \sim z$. Then $x, y \in A$ and $y, z \in B$ for some $A, B \in \mathcal{A}$. Since $y \in A$ and $y \in B$, $A \cap B \neq \emptyset$. By property (3) of a partition, $A = B$. Thus $x, z \in A$ and so $x \sim z$.

Problem 2. For each $r \in \mathbb{R}$, let $A_r = \{(x,y) \in \mathbb{R}^2 : x + y = r\}$ and let $\mathcal{A} = \{A_r : r \in \mathbb{R}\}$.

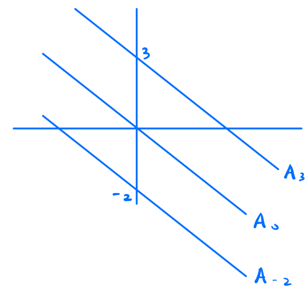
a. Make a sketch of a few elements of $\mathcal{A}$.

b. Prove that $\mathcal{A}$ is a partition of $\mathbb{R}^2$.

c. Consider the equivalence relation $\sim$ which $\mathcal{A}$ gives rise to.

1. Explain in geometric terms what it means for $(x,y) \sim (u,v)$.
2. What are the elements of the equivalence class of $(2,2)$?

(a) the elements of $\mathcal{A}$ are lines with slope -1 and y-intercept $r \in \mathbb{R}$:



(b) Let $r \in \mathbb{R}$. We claim A_r is nonempty. Indeed, $(0,r) \in A_r$. Since $A_r \subseteq \mathbb{R}^2$, it is clear $\bigcup_{r \in \mathbb{R}} A_r \subseteq \mathbb{R}^2$. On the other hand, we claim $\mathbb{R}^2 \subseteq \bigcup_{r \in \mathbb{R}} A_r$. Let $(x,y) \in \mathbb{R}^2$. Observe that $(x,y) \in A_{x+y}$. Thus $(x,y) \in A_r$ for some $r \in \mathbb{R}$ and so $(x,y) \in \bigcup_{r \in \mathbb{R}} A_r$. Thus $\bigcup_{r \in \mathbb{R}} A_r = \mathbb{R}^2$. Finally suppose A_r and A_s are given such that $A_r \cap A_s \neq \emptyset$. Since $A_r \cap A_s \neq \emptyset$, there exists $(x,y) \in \mathbb{R}^2$ such that $x+y=r$ and $x+y=s$. Thus $r=s$ and so $A_r = A_s$.

(c) The fact that $\{A_r : r \in \mathbb{R}\}$ is a partition of $\mathbb{R}^2$ means every point in $\mathbb{R}^2$ lies on some line of slope -1, and $(x,y) \sim (u,v)$ means (x,y) and (u,v) lie on the same line with slope -1. The equivalence class $E_{(2,2)}$ is the line $x+y=4$.