Chapter 12 Order in the Reals

Goal Explain (intuitively, semi-formally) how the real numbers $\mathbb{R}$ are constructed by "completing" the rationals $\mathbb{Q}$.

Definition A nonempty set $A \subseteq \mathbb{R}$ is said to be

① bounded above if there exists $M \in \mathbb{R}$ such that $x \leq M$ for all $x \in A$. We call M an upper bound of A.

② bounded below if there exists $m \in \mathbb{R}$ such that $x \geq m$ for all $x \in A$. We call m a lower bound of A.

③ bounded if it is bounded above and bounded below.

Examples Decide whether each of the following is bounded above, bounded below, or bounded. For those bounded above give 2 examples of upper bounds. Similarly give 2 examples of lower bounds for those that are bounded above.

① $A = \{x \in \mathbb{R} : x^2 \leq 5\} = [-\sqrt{5}, \sqrt{5}]$

② $B = \{x \in \mathbb{R} : x^3 < 5\} = (-\infty, 5^{1/3})$

③ $C = \{x \in \mathbb{N} : x \leq 5\} = \{0, 1, 2, 3, 4, 5\}$

④ $D = \{x \in \mathbb{Q} : x^2 \leq 2\} = [-\sqrt{2}, \sqrt{2}] \cap \mathbb{Q}$

⑤ $E = \{x \in \mathbb{Q} : x^2 < 2\} = (-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$

Definition A real number M is a **maximum** of a set $A \subseteq \mathbb{R}$ if $x \le M$ for all $x \in A$ and $M \in A$. A real number m is a **minimum** of A if $x \ge m$ for all $x \in A$ and $m \in A$.

Example Which sets in the previous example have a maximum? A minimum? If the max/min exists, what is it?

① $\max A = \sqrt{5}$, $\min A = -\sqrt{5}$

③ $\max C = 5$, $\min A = 0$

②, ④, ⑤ no max, no min

Definition Let $A \subseteq \mathbb{R}$ be a nonempty set that is bounded above. Then $U \in \mathbb{R}$ is said to be the **supremum** or **least upper bound** of A if

① $x \le U$ for all $x \in A$

② If M is an upper bound of A, then $U \le M$.

We denote the supremum of A by $\sup A$. If A is not bounded above, we let $\sup A = +\infty$.

Examples Which sets in the previous examples have a supremum? If it exists, what is it?

① $\sup A = \sqrt{5}$, ② $\sup B = 5^{1/3}$, ③ $\sup C = 5$, ④, ⑤ $\sup D = \sup E = \sqrt{2}$

Definition Let $A \subseteq \mathbb{R}$ be a nonempty set that is bounded below. Then $\ell \in \mathbb{R}$ is said to be the infimum or greatest lower bound of A if

① $x \geq \ell$ for all $x \in A$

② If m is a lower bound of A, then $\ell \geq m$.

We denote the infimum of A by $\inf A$.

Problem 1. Let $A \subseteq \mathbb{R}$ be a nonempty set. Answer the following true/false questions using definitions and intuition. No need to give a formal proof, but try to explain your reasoning for each.

a. An upper bound of A is an element of A.

b. The supremum of A is an element of A.

c. The set A has at least one maximum.

d. The set A has at most one maximum.

e. The set A has at least one upper bound.

f. The set A has at most one upper bound.

g. The set A has at least one supremum.

h. The set A has at most one supremum.

ⓐ false, 2 is a upper bound of $(0,1)$

ⓑ false, 1 is supremum of $(0,1)$

ⓒ false, $(0,1)$ has no maximum

ⓓ true

ⓔ false, $(0,\infty)$ has no upper bound

ⓕ false, $(0,1)$ has infinitely many upper bounds

Problem 2. Consider the interval $A = (3, 4]$.

a. Prove that A does not have a minimum.

b. Prove that $\inf A = 3$.

c. What can you say about $\max A$ and $\sup A$?

(a) Suppose A has a minimum m. Then $m \in A$ and $m \leq x$ for all $x \in A$. Since $m \in A$, $m > 3$. Consider $x_0 = \frac{m+3}{2}$, the midpoint between 3 and m. It is straightforward to see $x_0 \in A$ and so $m \leq x_0$. Thus $m \leq \frac{m+3}{2}$, which implies $2m \leq m+3$ which implies $m \leq 3$. But this contradicts that $m \in A$.

(b) Since $A = \{x \in \mathbb{R} : 3 < x \leq 4\}$, it is clear from definition of A that 3 is lower bound of A. Suppose m is an arbitrary lower bound of A. Then $m \leq x$ for all $x \in A$. We claim $m \leq 3$. Suppose $m > 3$. Then there must be some $x_0 \in A$ such that $3 < x_0 < m$, but this contradicts that m is a lower bound of A. Thus we have shown 3 is the greatest lower bound of A and hence $\inf A = 3$.

(c) $\max A = \sup A = 4$.