Problem 1. Let $f : A \to B$ be a bijection. Prove that f^{-1} is a bijection.

We first show f^{-1} is one-to-one.

Suppose $f^{-1}(y_1) = f^{-1}(y_2)$. Then taking f of both sides:

$$y_1 = f(f^{-1}(y_1)) = f(f^{-1}(y_2)) = y_2.$$

Next we show f^{-1} is onto. Let $a \in A$. We must show there exists $b \in B$ such that $f^{-1}(b) = a$. Let $b = f(a)$. Then $f^{-1}(b) = f^{-1}(f(a)) = a$.

Problem 2. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 - 5$. It can be shown that f is a bijection. Let's take this as given for this problem. Use algebra to determine f^{-1}. Note that using part 4 of today's theorem, you can check your work by computing $f \circ f^{-1}$ or $f^{-1} \circ f$.

$$y = x^3 - 5 \implies x^3 = y + 5$$
$$\implies x = (y+5)^{1/3}.$$

We claim $g(y) = (y+5)^{1/3}$ is $f^{-1}(y)$. To prove this is correct using Part 4 of Theorem 16.4, we should prove f is a bijection and check that $g \circ f = i_{\mathbb{R}}$ or $f \circ g = i_{\mathbb{R}}$.

Problem 3. We have previously shown that $f : \mathbb{Z} \to \mathbb{N}$ given by

$$f(n) = \begin{cases} 2n & n \geq 0 \\ -2n-1 & n < 0 \end{cases}$$

is a bijection. Find f^{-1}. Note that using part 4 of today's theorem, you can check your work by computing $f \circ f^{-1}$ or $f^{-1} \circ f$.

We claim $g(m) = \begin{cases} \frac{m}{2} & \text{if } m \text{ is even} \\ -\left(\frac{m+1}{2}\right) & \text{if } m \text{ is odd} \end{cases}$ is $f^{-1}(m)$.

To prove this is correct using Part 4 of Theorem 16.4,

we should prove f is a bijection and check that

$g \circ f = i_{\mathbb{R}}$ or $f \circ g = i_{\mathbb{R}}$.

Problem 4. Consider the functions $f : \mathbb{R} \setminus \{-2\} \to \mathbb{R} \setminus \{1\}$ and $g : \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{-2\}$ given by

$$f(x) = \frac{x-3}{x+2}, \quad g(x) = \frac{3+2x}{1-x}.$$

a. Calculate $f \circ g$ and $g \circ f$.

b. Can we use the previous part and part 4 of today's theorem to conclude anything?

(a) Doing algebraic simplification with fractions, we can check that $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$ for any x in the respective domains of g and f.

(b) Without checking that f is a bijection, it's not possible to make the conclusion that $g = f^{-1}$ using Part 4 of Theorem 16.4.

However, next time we'll prove that if $f : A \to B$ and $g : B \to A$ are functions such that $f \circ g = i_B$ and $g \circ f = i_A$, then they must be bijections and $f^{-1} = g$.