Chapter 16 Inverses, part II.

Theorem Let $f: A \to B$ and $g: B \to C$ be bijections. Then $g \circ f : A \to C$ is a bijection and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof We first show $g \circ f$ is one-to-one. Suppose $x_1, x_2 \in A$ and $(g \circ f)(x_1) = (g \circ f)(x_2)$. Notice that since g is bijective, it is one-to-one, which implies $f(x_1) = f(x_2)$. Similarly, f is bijective and thus one-to-one, so this implies $x_1 = x_2$. Next we show $g \circ f$ is onto.

Let $y \in C$. We must show there exists $x \in A$ such that $(g \circ f)(x) = y$. Since g is onto there exists $z \in B$ such that $g(z) = y$. Since f is onto there exists $x_0 \in A$ such that $f(x_0) = z$. Observe that

$$(g \circ f)(x_0) = g(f(x_0)) = g(z) = y.$$

Therefore $g \circ f$ is a bijection.

Observe that by associativity

$$\begin{aligned}(g \circ f) \circ (f^{-1} \circ g^{-1}) &= g \circ (f \circ f^{-1}) \circ g^{-1} \\ &= g \circ i_B \circ g^{-1} \\ &= g \circ g^{-1} = i_C\end{aligned}$$

By part (4) of the theorem from last time, we can conclude that this implies $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

As we proved the previous theorem we showed:

Theorem Let $f : A \to B$ and $g : B \to C$ be given functions.

① if f and g are one-to-one, then $g \circ f$ is one-to-one.

② if f and g are onto, then $g \circ f$ is onto.

The converses of the statements above are not true in general. But here a weaker statements that can be made:

Theorem Let $f : A \to B$ and $g : B \to C$ be given functions.

① if $g \circ f$ is one-to-one, then f is one-to-one

② if $g \circ f$ is onto, then g is onto

Proof ① Suppose $x_1, x_2 \in A$ and $f(x_1) = f(x_2)$. We must show $x_1 = x_2$. Notice that $g(f(x_1)) = g(f(x_2))$. Since $g \circ f$ is one-to-one, this implies $x_1 = x_2$.

② Let $y \in C$. We must show there exists $z \in B$ such that $g(z) = y$. Since $g \circ f$ is onto, there exists $x \in A$ such that $(g \circ f)(x) = y$. Let $z = f(x)$. Then $z \in B$ and $g(z) = g(f(x)) = y$.

We can use the previous theorem to show that $g : B \to A$ is invertible by proving there exists a function $f : A \to B$ such that $g \circ f = i_A$ and $f \circ g = i_B$. This addresses the final worksheet question from last time.

<u>Theorem</u> Let $f : A \to B$ and $g : B \to A$ be given functions. If $f \circ g = i_B$ and $g \circ f = i_A$, then g is bijective and $g^{-1} = f$.

<u>Proof</u> Since $g \circ f = i_A$ and i_A is onto, g is onto. Since $f \circ g = i_B$ and i_B is one-to-one, g is one-to-one. Therefore g is a bijection. By part 4 of the theorem from last time, $g^{-1} = f$.