Problem 1. Give a short proof that the set $\mathbb{Z}$ is a closed set in $\mathbb{R}$ using known results about open sets. State clearly any known results used.

We have shown (1) any open interval of the form (a,b) where $a,b \in \mathbb{R}$ is an open set, and

(2) any union of open sets is an open set.

Observe that $\mathbb{Z}^c = \bigcup_{n \in \mathbb{Z}} (n, n+1)$. Therefore, by

(1) and (2), $\mathbb{Z}^c$ is open. This means $\mathbb{Z}$ is closed.

Problem 2. Let $\{B_n : n \in \mathbb{Z}^+\}$ be a given collection of sets. Prove that $\mathcal{P}\left(\bigcap_{n=1}^{\infty} B_n\right) = \bigcap_{n=1}^{\infty} \mathcal{P}(B_n)$.

Let $A \in \mathcal{P}\left(\bigcap_{n=1}^{\infty} B_n\right)$. Then $A \subseteq \bigcap_{n=1}^{\infty} B_n$ which implies $A \subseteq B_n$ for all $n \in \mathbb{Z}^+$. Therefore, $A \in \mathcal{P}(B_n)$ for all $n \in \mathbb{Z}^+$, which implies $A \in \bigcap_{n=1}^{\infty} \mathcal{P}(B_n)$.

Thus $\mathcal{P}\left(\bigcap_{n=1}^{\infty} B_n\right) \subseteq \bigcap_{n=1}^{\infty} \mathcal{P}(B_n)$. Conversely, suppose $A \in \bigcap_{n=1}^{\infty} \mathcal{P}(B_n)$. Then $A \in \mathcal{P}(B_n)$ for all $n \in \mathbb{Z}^+$. This means $A \subseteq B_n$ for all $n \in \mathbb{Z}^+$. Thus, $A \subseteq \bigcap_{n=1}^{\infty} B_n$ and so $A \in \mathcal{P}\left(\bigcap_{n=1}^{\infty} B_n\right)$. Therefore we've proven the reverse inclusion and consequently the equality of the given sets.

Problem 3. If $a, b \in \mathbb{R}$, say that $a \sim b$ if and only if $a^k = b^k$ for some positive integer k. Prove that this is an equivalence relation on $\mathbb{R}$ or explain which properties hold and which fail. Repeat the question if the relation is changed to $a \sim b$ if and only if $a = b^k$ for some positive integer k.

This is an equivalence relation. For reflexivity, let $a \in \mathbb{R}$. Since $a^1 = a^1$, $a \sim a$. For symmetry, let $a, b \in \mathbb{R}$ and suppose $a \sim b$. Then $a^k = b^k$ for some $k \in \mathbb{Z}^+$. Therefore $b^k = a^k$, which means $b \sim a$. Finally, for transitivity, suppose $a, b, c \in \mathbb{R}$ and $a \sim b$ and $b \sim c$. Then $a^k = b^k$ and $b^\ell = c^\ell$ for some $k, \ell \in \mathbb{Z}^+$. Let $m = k\ell$. Then

$$a^m = (a^k)^\ell = (b^k)^\ell = (b^\ell)^k = (c^\ell)^k = c^m.$$

Therefore $a \sim c$.

The second relation is not. Symmetry fails. Note if $a = 4$, $b = 2$, then $a \sim b$ since $a = b^2$ but $b \not\sim a$. Reflexivity and transitivity have proofs similar to above.

Problem 4. Let $A_r = \{x \in \mathbb{R} : |x| = r\}$ for each $r \in \mathbb{R}$. Consider the collection $\mathcal{A} = \{A_r : r \in \mathbb{R}\}$. Is $\mathcal{A}$ a partition of $\mathbb{R}$? Explain which properties of a partition hold, if any, and which fail, if any.

This not a partition. Although $\bigcup_{r \in \mathbb{R}} A_r = \mathbb{R}$ and $A_r \cap A_s \neq \phi$ implies $A_r = A_s$, it is not the case that $A_r \neq \phi$ for all $r \in \mathbb{R}$. Notice $A_r = \phi$ whenever $r < 0$.

Problem 5. Find the supremum of $A = \{2 - 3/n : n \in \mathbb{Z}^+\}$ and prove that your value is correct. Does this set have a maximum or minimum? Find them if so.

We claim $\sup A = 2$. First note that since $\frac{3}{n} \geq 0$ for all $n \in \mathbb{Z}^+$, $2 - \frac{3}{n} \leq 2$ for all $n \in \mathbb{Z}^+$. Therefore 2 is an upper bound of A.

Next, suppose U is an upper bound of A. We claim $U \geq 2$. Suppose not. Then $U < 2$. By the Archimedean property of $\mathbb{R}$ (using $a = 3$, $b = 2 - U$) there exists $m \in \mathbb{Z}^+$ such that $2 - U > \frac{3}{m}$. Therefore $U < 2 - \frac{3}{m}$, which contradicts that U is an upper bound of A. Thus $U \geq 2$ indeed and we conclude $\sup A = 2$.

The set A does not have a maximum (since for every $x \in A$, there exists $y \in A$ such that $y > x$) but $\min A = -1$.

Problem 6. Let $f : A \to \mathbb{R}$ be given by $f(x) = \sqrt{x^2 - 4}$ where $A \subseteq \mathbb{R}$ is the largest set which is a valid domain of f.

a. Express A as an interval or union of intervals.

b. Prove that the range of f is $[0, \infty)$.

(a) $A = (-\infty, -2] \cup [2, \infty)$

(b) Let $y \in [0, \infty)$. We must show there exists $x \in A$ such that $f(x) = y$. Let $x = \sqrt{y^2+4}$. Then $x \in A$ since $y \geq 0$ implies $y^2 + 4 \geq 4$ which implies $\sqrt{y^2+4} \geq 2$. Moreover,

$$f(x) = f\left(\sqrt{y^2+4}\right) = \sqrt{\left(\sqrt{y^2+4}\right)^2 - 4} = y.$$

(Note $x = -\sqrt{y^2+4}$ would have worked too.)