Chapter 18 Mathematical Induction

A technique for proving a sequence of statements are all true.

Theorem (Principle of Mathematical Induction) For each $n \in \mathbb{Z}^+$ let $P(n)$ denote a logical statement. Suppose that

(1) $P(1)$ is true (called the base case)

(2) for each $n \in \mathbb{Z}^+$; $\underbrace{P(n)}_{\text{called the induction hypothesis}} \Rightarrow P(n+1)$. (called the induction step)

Then $P(n)$ is true for all $n \in \mathbb{Z}^+$.

Proof Suppose not. Then there is some collection of statements $\{P(n) : n \in \mathbb{Z}^+\}$ such that (1) and (2) are true but $P(n_0)$ is false for some $n_0 \in \mathbb{Z}^+$.

Let $A = \{n \in \mathbb{Z}^+ : P(n) \text{ is false}\}$. Since $n_0 \in A$, A is non-empty. By the well-ordering principle of $\mathbb{N}$, A has a minimum element, which call m. By condition (1), $m \neq 1$ and so $m \geq 2$. By (2)

$P(m-1) \Rightarrow P(m)$. But this is a contradiction since $P(m-1)$ is true but $P(m)$ is false.

Example Use the Principle of Mathematical Induction to prove that

(1) $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{for all } n \in \mathbb{Z}^+$$

(2) $$\underbrace{\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}}_{\text{the formula is } P(n)}$$

Proof of (2) We proceed by induction on n.

The base case (when $n=1$) is the formula

$$1^2 = \frac{1(1+1)(2(1)+1)}{6}$$

which is clearly true by simplifying the right side.

For the induction step, let $n \in \mathbb{Z}^+$ and suppose the formula

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

is true. We must use this to show the formula holds when n is replaced by $n+1$. Observe that

$$\sum_{k=1}^{n+1} k^2 = 1^2 + 2^2 + \cdots + n^2 + (n+1)^2$$

$$= \sum_{k=1}^{n} k^2 + (n+1)^2$$

$$= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \quad \text{by induction hypothesis}$$

$$= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}$$

$$= \frac{(n+1)\left(n(2n+1)+6(n+1)\right)}{6}$$

$$= \frac{(n+1)(2n^2+7n+6)}{6}$$

$$= \frac{(n+1)(n+2)(2n+3)}{6}$$

$$= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}.$$

Therefore the formula holds for all $n \in \mathbb{Z}^+$.

Example Prove that for all $n \in \mathbb{N}$, $3 \mid (4^n - 1)$.

Notice that sometimes we want to prove a collection of statements where the base case is $n = 0$.

Proof We proceed by induction on n. The base case, where $n = 0$, states $3 \mid (4^0 - 1)$. This is clearly true.

For the induction step, let $n \in \mathbb{N}$ and suppose $3 \mid (4^n - 1)$.

We must show $3 \mid (4^{n+1} - 1)$. That is, we must show $3k = 4^{n+1} - 1$ for some $k \in \mathbb{Z}$. Since $3 \mid (4^n - 1)$, we have $3m = 4^n - 1$ for some $m \in \mathbb{Z}$. Therefore

$$4^{n+1} - 1 = 4 \cdot 4^n - 1$$

$$= 4(3m+1) - 1$$

$$= 12m + 3$$

$$= 3(4m+1)$$

$$= 3k$$

where $k = 4m+1 \in \mathbb{Z}$.

Problem 1. Use induction to prove that for every $n \in \mathbb{Z}^+$

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

The base case (when $n=1$) is $1 = \frac{1(1+1)}{2}$. This is clearly true. Suppose $n \in \mathbb{Z}^+$ and suppose $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

Then $\displaystyle\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + n+1$

$$= \frac{n(n+1)}{2} + n+1$$

$$= \frac{n^2 + n + 2n+2}{2}$$

$$= \frac{n^2+3n+2}{2}$$

$$= \frac{(n+1)(n+2)}{2}.$$

$$= \frac{(n+1)((n+1)+1)}{2}.$$

Problem 2. Let $n \geq 1$ be an integer and let $x_1, \ldots, x_n \in \mathbb{R}$. Use induction to prove that

$$|x_1 + \ldots + x_n| \leq |x_1| + \cdots |x_n|.$$

The base case (when $n = 1$) is $|x_1| \leq |x_1|$, which is clearly true. Suppose $n \in \mathbb{Z}^+$ and suppose $|x_1 + \cdots + x_n| \leq |x_1| + \cdots + |x_n|$.

Observe that

$$\begin{aligned} |x_1 + \cdots + x_n + x_{n+1}| &= |(x_1 + \cdots + x_n) + x_{n+1}| \\ &\leq |x_1 + \cdots + x_n| + |x_{n+1}| \qquad (1) \\ &\leq |x_1| + \cdots + |x_n| + |x_{n+1}| \qquad (2) \end{aligned}$$

where (1) is by the triangle ineq. and (2) is by the induction hypothesis.

Problem 3. Let $x \in (-1, \infty)$. Use induction to prove Bernoulli's inequality: $(1+x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

The base case (when $n = 0$) is $(1+x)^0 \geq 1$. This is clearly true. Suppose $n \in \mathbb{N}$ and $(1+x)^n \geq 1 + nx$.

Observe that

$$\begin{aligned} (1+x)^{n+1} &= (1+x)^n \cdot (1+x) \\ &\geq (1+nx)(1+x) \qquad (1) \\ &= 1 + nx + x + nx^2 \\ &= 1 + (n+1)x + nx^2 \\ &\geq 1 + (n+1)x \qquad (2) \end{aligned}$$

where (1) is due to the induction hypothesis and (2) is due to $nx^2 \geq 0$.

Problem 4. Use induction to prove that 8 divides $5^{2n} - 1$ for all $n \in \mathbb{N}$.

The base case $(n=0)$ is $8 \mid (5^0 - 1)$ which is true.

Suppose $n \in \mathbb{N}$ and suppose $8 \mid (5^{2n} - 1)$. Then $8m = 5^{2n} - 1$ for some $m \in \mathbb{Z}$. Observe that

$$\begin{aligned} 5^{2(n+1)} - 1 &= 5^{2n} \cdot 5^2 - 1 \\ &= (8m+1) \cdot 5^2 - 1 \\ &= 8(25m) + 24 \\ &= 8(25m+3) \\ &= 8k \end{aligned}$$

where $k = 2m+3 \in \mathbb{Z}$. Thus $8 \mid (5^{2(n+1)} - 1)$.