Chapter 18 More induction

Theorem (Second principle of mathematical induction, also known as Strong Induction) For each $n \in \mathbb{Z}^+$, let $P(n)$ denote a logical statement. Suppose that

(1) $P(1)$ is true (base case)

(2) for each $n \in \mathbb{Z}^+$, $\underbrace{P(1), \dots, P(n)}_{\text{induction hypothesis}} \Rightarrow P(n+1)$. (induction step)

Then $P(n)$ is true for all $n \in \mathbb{Z}^+$.

Example Define a sequence $(a_n)_{n=1}^{\infty}$ by $a_1 = 1$, $a_2 = 3$ and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 3$. Prove that $a_n < \left(\frac{7}{4}\right)^n$ for all $n \in \mathbb{Z}^+$.

Proof We proceed by strong induction. The base case $(n=1)$ is $1 < \left(\frac{7}{4}\right)^1$ which is clearly true. Let $n \in \mathbb{Z}^+$ and suppose $a_k < \left(\frac{7}{4}\right)^k$ for all $k = 1, \dots, n$. Observe that

$$
\begin{aligned}
a_{n+1} &= a_n + a_{n-1} \\
&< \left(\frac{7}{4}\right)^n + \left(\frac{7}{4}\right)^{n-1} \qquad \text{by induction hypothesis}
\end{aligned}
$$

$$= \left(\frac{7}{4}\right)^n \left(1 + \frac{4}{7}\right)$$

$$= \left(\frac{7}{4}\right)^n \left(\frac{11}{7}\right)$$

$$< \left(\frac{7}{4}\right)^n \left(\frac{7}{4}\right)$$

$$= \left(\frac{7}{4}\right)^{n+1}.$$

Therefore, by strong induction, the inequality holds for all $n \in \mathbb{Z}^+$.

Example Consider the sequence $(a_n)_{n=1}^{\infty}$ defined by

$a_1 = 1$, $a_2 = 1$, $a_n = a_{n-1} + a_{n-2} - n + 4$ for all $n \geq 3$.

Prove that $a_n \geq n$ for all $n \geq 3$.

Proof We proceed by strong induction. The base case is $a_3 \geq 3$ which is true since $a_3 = 1 + 1 - 3 + 4 = 3$.

Suppose $n \in \mathbb{Z}^+$ such that $n \geq 3$ and suppose $a_k \geq k$ for all $k = 3, \ldots, n$. Observe that

$$\begin{aligned}
a_{n+1} &= a_n + a_{n-1} - (n+1) + 4 \\
&\geq n + (n-1) - (n+1) + 4 \quad \text{by induction hypothesis} \\
&= n + 2 \\
&> n + 1
\end{aligned}$$

Therefore the inequality holds for all n by strong induction.

Problem 1 $n^2 \geq n+1$, $n \geq 2$

base case: $n=2$

$$(2)^2 \geq 2+1$$
$$4 \geq 3 \checkmark$$

induction: let $n \in \mathbb{N}$, suppose $n^2 \geq n+1$

w.t.s $(n+1)^2 \geq n+2$

Observe that
$$\begin{aligned}(n+1)^2 &= (n+1)(n+1)\\ &= n^2+2n+1\\ &\geq n+1+2n+1 \quad (*)\\ &= n+2+2n\\ &\geq n+2\end{aligned}$$

So by induction, $n^2 \geq n+1$ for all $n \geq 2$ □

$(*)$ is true by ind. hyp

Problem 2 $n! > n^2$, $n \geq 4$

Base case: $(4! = 24) > (4^2 = 16)$

Inductive step: Let $n \in \mathbb{N}$. Assume $n! > n^2$. WTS $(n+1)! > (n+1)^2$.

Observe that
$$\begin{aligned}(n+1)! &= (n+1)n!\\ &> (n+1)n^2 \quad \text{by induction hypothesis}\\ &\geq (n+1)(n+1) \quad \text{by problem 1}\\ &= (n+1)^2.\end{aligned}$$

Thus we have shown that $(n+1)! > (n+1)^2$ & the proof by induction is finished. □

Problem 3 Prove that $\sum_{k=0}^{n} 2^k = 2^{n+1}-1$ for any $n \in \mathbb{Z}^+$

proof: We will proceed by induction.

The base case $n=0$ says that $2^0 = 2^{0+1}-1 = 1$. That is clearly true.

For the induction step.

Let $n \in \mathbb{Z}^+$ and suppose that $\sum_{k=0}^{n} 2^k = 2^{n+1}-1$

Our goal is to show that $\sum_{k=0}^{n+1} 2^k = 2^{(n+1)+1}-1 = 2^{n+2}-1$

Observe that
$$\begin{aligned}\sum_{k=0}^{n+1} 2^k &= \sum_{k=0}^{n} 2^k + 2^{n+1}\\ &= 2^{n+1}-1+2^{n+1} \quad \text{(by induction hypothesis)}\\ &= 2^1 \cdot 2^{n+1}-1\\ &= 2^{n+2}-1\end{aligned}$$

By induction. $\sum_{k=0}^{n} 2^k = 2^{n+1}-1$ for any $n \in \mathbb{Z}^+$

P4 a) $T_1 = 1+\sum_{k=0}^{0} T_k = 2$

$T_2 = 1+\sum_{k=0}^{1} T_k = 1+T_0+T_1 = 4$

$T_3 = 1+\sum_{k=0}^{2} T_k = 1+T_0+T_1+T_2 = 8$

$T_4 = 1+\sum_{k=0}^{3} T_k = 1+T_0+T_1+T_2+T_3 = 16$

b) We will proceed by strong induction to prove $T_n = 2^n$.
For the base case, $n=0$. So $T_0 = 1 = 2^0$, and the equation holds.
For the induction step, let $n \in \mathbb{N}$ and $T_k = 2^k$ for all $k = 0, \ldots, n$.
We will show $T_{n+1} = 2^{n+1}$ for all $n \in \mathbb{N}$. Observe that

$$\begin{aligned}T_{n+1} &= 1+T_0+T_1+\ldots+T_n\\ &= 1+2^0+2^1+\ldots+2^n && \text{by induction hypothesis}\\ &= 1+2^{n+1}-1 && \text{by Problem 3}\\ &= 2^{n+1}.\end{aligned}$$

By strong induction the equation holds for all $n \in \mathbb{N}$.

Problem 5

We proceed by strong induction. For the base case, $n=0$, we have $\frac{a^0-b^0}{a-b} = 0 = F_0$. For the induction step, suppose $n \in \mathbb{N}$ and $F_k = \frac{a^k - b^k}{a-b}$ for all $k=0,\dots,n$.

Observe that

$$F_{n+1} = F_n + F_{n-1}$$

$$= \frac{a^n - b^n}{a-b} + \frac{a^{n-1} - b^{n-1}}{a-b} \quad \text{by ind. hypothesis}$$

$$= \frac{a^n + a^{n-1} - (b^n + b^{n-1})}{a-b}$$

$$= \frac{a^{n-1}(a+1) - b^{n-1}(b+1)}{a-b}$$

$$= \frac{a^{n-1}\cdot a^2 - b^{n-1}\cdot b^2}{a-b} \quad \text{since } a,b \text{ are solutions to } x^2 = x+1$$

$$= \frac{a^{n+1} - b^{n+1}}{a-b}.$$

Therefore by strong induction, the formula holds for all $n \in \mathbb{N}$.