Chapter 19 Sequences

Def A <u>sequence</u> of real numbers is a function $f : \mathbb{N} \to \mathbb{R}$ (or sometimes we use $\mathbb{Z}^+$ for the domain).

We often let $x_n = f(n)$ for each $n \in \mathbb{N}$ (or $n \in \mathbb{Z}^+$)

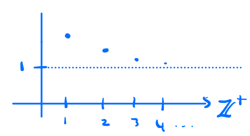
and denote the sequence as $(x_n)_{n=0}^{\infty}$ (or $(x_n)_{n=1}^{\infty}$)

or simply (x_n) when there is no ambiguity about the starting value of n.

Remark x_n is a single element of the sequence (the n^{th} element) and (x_n) is the whole sequence.

Examples

① $x_n = 1 + \frac{1}{n}$ for $n \in \mathbb{Z}^+$



② $x_n = 1 - (-1)^n$ for $n \in \mathbb{N}$ $\quad x_0 = 0,\ x_1 = 2,\ x_2 = 0,\ x_3 = 2, \ldots$

③ $x_n = \dfrac{n}{n+1}$ for $n \in \mathbb{N}$ $\quad x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = \frac{2}{3},\ x_3 = \frac{3}{4}, \ldots$

④ $x_n = \dfrac{n^2+1}{1-n}$ for $n = 2, 3, \ldots$ $\quad x_2 = -5,\ x_3 = -5,\ x_4 = -\frac{17}{3}, \ldots$

Terminology Let $(x_n)_{n=0}^{\infty}$ be a given sequence and let $S = \{x_n : n \in \mathbb{N}\}$ be its set of elements.

We can say (x_n) is <u>bounded above</u>, <u>bounded below</u>, or <u>bounded</u> and we mean S is bounded above, bounded below, or bounded. Similarly, by an <u>upper/lower bound</u> of (x_n) we mean an upper/lower bound of S. And $\inf(x_n) = \inf S$ and $\sup(x_n) = \sup S$.

<u>Example</u> Let (x_n) be a sequence. Prove that (x_n) is bounded if and only if there exists $M \in \mathbb{R}$ such that $|x_n| \le M$ for all n.

<u>Proof</u> ($\Longrightarrow$) Suppose (x_n) is bounded. Then it is bounded above, which implies there exists $U \in \mathbb{R}$ such that $x_n \le U$ for all n. Similarly, it is bounded below, which implies there exists $L \in \mathbb{R}$ such that $x_n \ge L$ for all n. Let $M = \max\{|U|, |L|\}$.

Then $x_n \le U \le |U| \le M$ for all n. Similarly, $x_n \ge L \ge -|L| \ge -M$ for all n. Thus $|x_n| \le M$ for all n.

The ($\Longleftarrow$) proof is left as an exercise.

<u>Example</u> Let (x_n) and (y_n) be bounded sequences and let $M_x = \sup(x_n)$ and $M_y = \sup(y_n)$. Let $z_n = x_n + y_n$ for all n. Prove $\sup(z_n) \le M_x + M_y$ but it's not necessarily true that $\sup(z_n) = M_x + M_y$.

Proof Since $z_n = x_n + y_n \le M_x + M_y$ for all n, $M_x + M_y$ is an upper bound for (z_n). Therefore $\sup(z_n) \le M_x + M_y$ since $\sup(z_n)$ is the least upper bound of (z_n).

Consider $(x_n) = (1, 2, 2, \ldots)$ and $(y_n) = (-1, -2, -2, \ldots)$. Then $(z_n) = (0, 0, 0, \ldots)$. So $\sup(z_n) = 0$ and $M_x = 2$, $M_y = -1$, $M_x + M_y = 1 \ne \sup(z_n)$.

Problem 1. State the infimum of each of the following sequences. No justification needed.

a. $x_n = 1/n$ for $n \in \mathbb{Z}^+$

b. $x_n = n^2$ for $n \in \mathbb{N}$

c. $x_n = n/(n+1)$ for $n \in \mathbb{N}$

d. $x_n = (-1)^n/(n^2+1)$ for $n \in \mathbb{N}$

(a) 0

(b) 0

(c) $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$ $\quad 0$

(d) $1, \frac{-1}{2}, \frac{1}{5}, \frac{-1}{10}, \frac{1}{17}, \frac{-1}{26}, \ldots$ $\quad -\frac{1}{2}$

Problem 3. Let (x_n) be a sequence that is bounded below. Let $L = \inf(x_n)$. Define $y_n = -x_n$ for all n. Prove that (y_n) is bounded above, make a conjecture about the value of $\sup(y_n)$, and prove your conjecture.

Since $x_n \ge L$ for all n, $y_n = -x_n \le -L$ for all n. Therefore (y_n) is bounded above by $-L$. We claim $\sup(y_n) = -L$. Let $U \in \mathbb{R}$ be an arbitrary upper bound of (y_n). Then since $y_n \le U$ for all n,

$x_n = -y_n \geq -U$ for all n. Therefore, $-U$ is a lower bound of (x_n) which implies $L \geq -U$ since L is the greatest lower bound of (x_n). Therefore $-L \leq U$ which implies $-L$ is the least upper bound of (y_n).

Problem 2. A sequence (x_n) is **increasing** if $x_n \leq x_{n+1}$ for all n and **decreasing** if $x_n \geq x_{n+1}$ for all n. It is **strictly increasing** if $x_n < x_{n+1}$ for all n and **strictly decreasing** if $x_n > x_{n+1}$ for all n. For each of the following give an example of a sequence that meets the given constraints or explain why it is not possible.

a. A bounded sequence. Find an upper bound and a lower bound.

b. A sequence that is bounded below but not bounded above. Find a lower bound. Does your example need to be an increasing sequence?

c. A sequence that is bounded above but not bounded below. Find an upper bound.

d. An increasing sequence that is neither bounded above nor below.

e. A strictly increasing bounded sequence.

f. A strictly decreasing sequence that is bounded above but not below.

g. A sequence that is neither strictly increasing nor strictly decreasing.

(a) $a_n = (-1)^n \quad n \in \mathbb{N}$

(b) $a_n = n$, $n \in \mathbb{N}$, lower bound 0, need not be increasing

(c) $a_n = -n$, $n \in \mathbb{N}$ upper bound 0

(d) not possible (increasing sequences are always bounded below with the first term serving as lower bound)

(e) $a_n = 1 - \frac{1}{n}$, $n \in \mathbb{Z}^+$

(f) $a_n = -n$, $n \in \mathbb{N}$

(g) $a_n = (-1)^n$, $n \in \mathbb{N}$