Chapter 3 Contrapositive, converse, intro. to proof writing

General outline of a proof:

- State assumptions
- State what you'll prove
- Do a calculation, algebraic manipulation, or series of logical deductions.
- Write a conclusion statement.

Some writing guidelines:

- Use complete sentences with punctuation.
- Avoid starting sentences with numbers, variables, or other mathematical symbols; use words instead.
- Put important calculation on its own line and otherwise write in paragraphs

Def The contrapositive of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$.

Remark The contrapositive $\neg Q \Rightarrow \neg P$ is equivalent to $P \Rightarrow Q$:

P	Q	$P \Rightarrow Q$	$\neg Q$	$\neg P$	$\neg Q \Rightarrow \neg P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Example Let x be an integer. Write the contrapositive of "If x^2 is odd, then x is odd."

"If x is even then x^2 is even"

Def An integer x is *even* if there is some integer n such that $x = 2n$. It is *odd* if there is some integer n such that $x = 2n+1$.

Let's try to prove "if x is even then x^2 is even"

Proof outline:

ASSUME: - x is an integer
- x is even

GOAL: show x^2 is even
(show $x^2 = 2m$ for some integer m)

STEPS: x is even $\Rightarrow$ $\exists$ an integer n such that
"there exists" $x = 2n$

$$\Rightarrow x^2 = (2n)^2$$
$$= 4n^2$$
$$= 2(2n^2)$$

Proof Let x be an even integer. Then $x = 2n$ for some integer n. We aim to show that x^2 is even by showing there exists an integer m such that $x^2 = 2m$. Observe that

$$\begin{aligned} x^2 &= (2n)^2 \\ &= 4n^2 \\ &= 2(2n^2). \end{aligned}$$

notice the period to end the sentence with punctuation

Thus, since $m = 2n^2$ is an integer and $x^2 = 2m$, x^2 is even.

Def The converse of $P \Rightarrow Q$ is $Q \Rightarrow P$

Remark $Q \Rightarrow P$ is not equivalent to $P \Rightarrow Q$:

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Example Prove the converse of

"If x^2 is odd, then x is odd."

Converse is "If x is odd, then x^2 is odd."

Proof outline

ASSUME : x is an odd integer

GOAL : show x^2 is odd (show $x^2 = 2m+1$ for some m)

STEPS : x odd $\Rightarrow x = 2n+1$ for some n

$$\begin{aligned} \Rightarrow x^2 &= (2n+1)^2 \\ &= 4n^2+4n+1 \\ &= 2(2n^2+2n)+1 \end{aligned}$$

Proof Let x be an odd integer. Then $x = 2n+1$ for some integer n. We aim to show that x^2 is odd by showing there exists an integer m such that $x^2 = 2m+1$. Observe that

$$\begin{aligned} x^2 &= (2n+1)^2 \\ &= 4n^2+4n+1 \\ &= 2(2n^2+2n)+1 \end{aligned}$$

Thus, since $m = 2n^2+2n$ is an integer and $x^2 = 2m+1$, x^2 is odd.

Problem 1. For each of the following statements, write out the contrapositive and converse statements. Discuss whether either is true.

a. If you are the President of the United States, then you live in a white house.

b. If you are going to bake a souffle, then you need eggs.

c. If x is a real number, then x is an integer.

d. If x is a real number, then $x^2 < 0$.

(a) contrapositive : If you don't live in a white house, then you are not the POTUS. (true)

converse : If you live in a white house, then you are the POTUS. (false)

(b) contrapositive: If you don't need eggs, then you're not going to bake a soufflé. (true)

converse: If you need eggs, then you're going to bake a soufflé. (false)

(c) contrapositive: If x is not an integer, then x is not a real number (false)

converse: If x is an integer, then x is a real number. (true)

(d) contrapositive: if $x^2 \geq 0$, then x is not a real number, (false)

converse: if $x^2 < 0$, then x is a real number (false)

Problem 2. Consider the statement "if x is odd, then $x^2 + 3x + 1$ is odd."

a. State the contrapositive.

b. Prove either the statement itself or its contrapositive.

c. State the converse.

d. Explain why the converse is not true by giving a counterexample.

(a) If $x^2 + 3x + 1$ is even, then x is even.

(b) We prove the statement itself. Let x be an odd number. Then $x = 2n+1$ for some integer n.

Therefore
$$\begin{aligned} x^2 + 3x + 1 &= (2n+1)^2 + 3(2n+1) + 1 \\ &= 4n^2 + 4n + 1 + 6n + 3 + 1 \\ &= 4n^2 + 10n + 4 + 1 \\ &= 2(2n^2 + 5n + 2) + 1. \end{aligned}$$

Thus $x^2 + 3x + 1 = 2m + 1$ where $m = 2n^2 + 5n + 2$ is an integer, which means $x^2 + 3x + 1$ is odd.

(c) If $x^2 + 3x + 1$ is odd, then x is odd.

(d) Consider $x = 0$. Then $x^2 + 3x + 1 = 1$ is odd but $x = 0$ is even.

Problem 3. Consider the statement "if x and y are real numbers, then $x^2 + y^2 \geq 2xy$." This is a true statement and our goal in this problem will be to write a proof. It might feel hard to start because the assumptions (that x and y are real numbers) are so open ended. Sometimes it helps to do some scratch work and generate ideas by manipulating the conclusion and working backwards from the conclusion. Try doing that here and then write your proof. Make sure your proof starts from the given assumptions and then reaches the desired conclusion.

Scratch work: $x^2 + y^2 \geq 2xy \iff x^2 - 2xy + y^2 \geq 0$

$$\iff \underbrace{(x-y)^2 \geq 0}_{\text{always true for any real numbers } x, y}$$

Proof Let x and y be real numbers. Since the square of any real number is non-negative, $(x-y)^2 \geq 0$. Expanding the left hand side, this implies

$$x^2 - 2xy + y^2 \geq 0,$$

which implies $x^2 + y^2 \geq 2xy$.

Problem 4. Suppose x and y are positive real numbers. Prove that if $x < y$ then $x^2 < y^2$. Use an approach like in the previous problem.

Scratch work: $x^2 < y^2 \iff y^2 - x^2 > 0$

$$\iff \underbrace{(y-x)}_{\text{assumed to be } > 0}\underbrace{(y+x)}_{y>0,\ x>0 \implies y+x>0} > 0$$

Proof Let x and y be positive real numbers such that $x < y$. Since x and y are positive, $x > 0$ and $y > 0$, which implies $y + x > 0$. Since $x < y$, we have $y - x > 0$. Therefore since the product of positive numbers is positive, we have $(y+x)(y-x) > 0$. Expanding the left hand side gives $y^2 - x^2 > 0$, which implies $x^2 < y^2$.