Chapter 4 Set notation and quantifiers

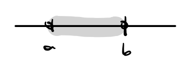
Let's start with introducing notation:

- the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$ some authors don't include 0
- the integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- the rational numbers $\mathbb{Q} = \{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \}$

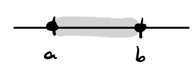
  ":" "such that" "$\in$" "element of"

- the real numbers $\mathbb{R}$ (informally: the rational number along with values along the number line that you get when taking limits)

- open interval $(a, b) = \{ x \in \mathbb{R} : a < x < b \}$



- closed interval $[a, b] = \{ x \in \mathbb{R} : a \leq x \leq b \}$



Quantifiers Many mathematical statements the quantifying phrases "for all" or "there exists," which have shorthand notation $\forall$ and $\exists$ respectively.

Examples Rephrase the following using quantifiers:

① if $x \in \mathbb{R}$, then $x^2 \geq 0$ "for all $x \in \mathbb{R}$, $x^2 \geq 0$"
"$\forall x \in \mathbb{R},\ x^2 \geq 0$"

② the equation $x^2 + 2x = 15$ has a real number solution. "$\exists\ x \in \mathbb{R}$ such that $x^2 + 2x = 15$"

③ every real number has a square root that is a real number "$\forall\ x \in \mathbb{R},\ \sqrt{x} \in \mathbb{R}$"

or "$\forall\ x \in \mathbb{R}, \exists\ y \in \mathbb{R}$ such that $y = \sqrt{x}$"

Questions How do we negate "Every cow is black"?

Valid: "Not every cow is black"

Equivalent: "There is a cow that is not black"

How do we negate

"There is someone who speaks Spanish in class"?

Valid: "No one speaks Spanish in class"

Equivalent: "Every person in class does not speak Spanish"

Negating $\forall, \exists$:

$$\neg (\forall x, P(x)) \iff \exists x, \neg P(x)$$

"it is not the case that for all x, $P(x)$ is true" "there exists x such that $P(x)$ is not true"

$$\neg (\exists x, P(x)) \iff \forall x, \neg P(x)$$

"it is not the case that there exists x such that $P(x)$ is true" "for all x, $P(x)$ is not true"

Example Express the following in natural language, negate it symbolically, and decide whether it's true.

① $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y^3 = x$

"Every real number x has a corresponding cube root y." (true)

negation: $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, y^3 \neq x$.

② $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, y^3 = x$

"There is a real number x which is the cube of every real number." (false)

negation: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y^3 \neq x$.

Problem 1. Write the following statements symbolically.

a. If x is a real number, then it is an integer.

b. There is an integer x that is non-negative.

c. The equation $x^2 + 1 = 0$ has a rational solution.

d. For every real number x, there is a real number y such that $x < y$.

e. There is a real number y such that $x < y$ for all $x \in \mathbb{R}$.

f. If x is a positive real number, then $x > 4$ or $x < 6$.

(a) $\forall x \in \mathbb{R},\ x \in \mathbb{Z}$

(b) $\exists x \in \mathbb{Z},\ x \geq 0$

(c) $\exists x \in \mathbb{Q},\ x^2+1=0$

(d) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R},\ x<y$

(e) $\exists y \in \mathbb{R}, \forall x \in \mathbb{R},\ x<y$

(f) $\forall x \in (0,\infty),\ x>4$ or $x<6$

Problem 2. Negate the statements in the previous problem. Decide which is true for each: the original statement or its negation.

(a) $\exists x \in \mathbb{R},\ x \notin \mathbb{Z}$ true

"There exists a real number that is not an integer."

(b) $\forall x \in \mathbb{Z},\ x<0$ false

"Every integer is negative"

(c) $\forall x \in \mathbb{Q},\ x^2+1 \neq 0$ true

"No rational number satisfies $x^2+1=0$"

(d) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R},\ x \geq y$ false

"There is a largest real number x."

(e) $\forall y \in \mathbb{R}, \exists x \in \mathbb{R},\ x \geq y$ true

"Every real number y has at least one real number x that is at least as big as y."

(f) $\exists x \in \mathbb{R},\ x \leq 4$ and $x \geq 6$ false

"There is a real number that lives in both the interval $[-\infty, 4]$ and the interval $[6, \infty)$."

Problem 3. Consider the following statements. Rephrase them in words and decide whether either is true.

a. $\forall n \in \mathbb{Z}, \exists m \in \mathbb{Z}, m = n + 5$

b. $\exists m \in \mathbb{Z}, \forall n \in \mathbb{Z}, m = n + 5$

(a) Every integer n has a corresponding integer m that satisfies $m = n+5$. true

(b) There is an integer m that is 5 plus any integer. false.