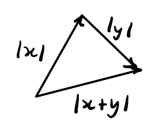
Chapter 5 Absolute value, inequalities and proofs by cases.

Def For any $x \in \mathbb{R}$, the absolute value of x is given by
$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Intuition $|x|$ measures the distance from x to 0.

Theorem Let $x, y \in \mathbb{R}$. Then

1. $|xy| = |x||y|$
2. $|x+y| \leq |x| + |y|$ (called the triangle inequality)
3. $|x-y| \leq |x| + |y|$



Proof of 1 We proceed by cases.

Consider the case when $x > 0$ and $y > 0$. Then $xy > 0$ and $x = |x|$ and $y = |y|$. Therefore

$$|xy| = xy = |x||y|.$$

Next consider the case when $x < 0$ and $y < 0$. Then $xy > 0$ and $-x = |x|$ and $-y = |y|$. Therefore

$$|xy| = xy = (-x)(-y) = |x||y|.$$

Third, we consider the case when $x < 0$ and $y > 0$. Then $xy < 0$, $-x = |x|$, and $y = |y|$. Thus

$$|xy| = -xy = (-x)(y) = |x||y|.$$

Finally we note that the case when $x > 0$ and $y < 0$ is the same as the third case.

Here are some basic rules about inequalities we should know and use freely

Facts Let $a, b, c \in \mathbb{R}$

1. if $a \le b$ and $b \le c$, then $a \le c$
2. if $a \le b$, then $a \pm c \le b \pm c$
3. if $a \le b$ and $c \ge 0$, then $ac \le bc$
4. if $a \le b$, then $-a \ge -b$.
5. let $a \ge 0$. Then $|x| \le a$ if and only if $-a \le x \le a$.

Proof of 5 Suppose $|x| \le a$. We aim to show $-a \le x \le a$.

Consider the case when $x \ge 0$. Then

$$\begin{aligned} -a &\le x \\ &= |x| \\ &\le a. \end{aligned}$$

Therefore $-a \leq |x| \leq a$.

Next, with the case when $x < 0$, we have

$x < 0 \leq a$. Moreover, $-x = |x| \leq a$, which implies

$x \geq -a$. Thus $-a \leq x \leq a$ in both cases.

Now suppose $-a \leq x \leq a$. We now show $|x| \leq a$

again proceeding by cases. Suppose $x \geq 0$. Then

$|x| = x \leq a$. Next suppose $x < 0$. Notice that

$-a \leq x$ implies $a \geq -x = |x|$. Therefore $|x| \leq a$ in

both cases.

Problem 2. Sometimes it is useful to use the old trick of adding zero: for any real numbers x and a we have $x = x + a - a$. Use this trick along with the triangle inequality to prove that if $x \in \mathbb{R}$ and if $|x - 3| \leq 1$ then $|x^2 - 9| \leq 7$.

Proof Let $x \in \mathbb{R}$ and suppose $|x - 3| \leq 1$. Observe

that
$$
\begin{aligned}
|x^2 - 9| &= |(x-3)(x+3)| \\
&= |x-3||x+3| \\
&\leq |x+3| \\
&= |x - 3 + 3 + 3| \\
&= |x - 3 + 6| \\
&\leq |x-3| + 6 \\
&\leq 7.
\end{aligned}
$$

Problem 3. Let $x \in \mathbb{R}$. Prove that if $|x+1| \leq 2$ then $|(x+1)(x^3-2)| \leq 58$.

Suppose $|x+1| \leq 2$. Observe that

$$
\begin{aligned}
|(x+1)(x^3-2)| &= |x+1||x^3-2| \\
&\leq 2|x^3-2| \\
&\leq 2(|x|^3+2) \\
&= 2|x|^3+4 \\
&\leq 2|x+1-1|^3+4 \\
&\leq 2(|x+1|+1)^3+4 \\
&\leq 2(2+1)^3+4 \\
&= 58
\end{aligned}
$$

Problem 4. Express the following sets as open or closed intervals using interval notation.

a. $\{x \in \mathbb{R} : |x| < 5\}$

b. $\{x \in \mathbb{R} : |x| \leq 1\}$

c. $\{x \in \mathbb{R} : |x-1| < 3\}$

d. $\{x \in \mathbb{R} : |x+2| \leq 8\}$

(a) $(-5, 5)$

(b) $[-1, 1]$

(c) $|x-1| < 3 \iff -3 < x-1 < 3$

$\iff -2 < x < 4$ $\qquad (-2, 4)$

(d) $|x+2| \leq 8 \iff -8 \leq x+2 \leq 8$

$\iff -10 \leq x \leq 6$ $\qquad [-10, 6]$

Alternate proof of the triangle inequality

Let $x, y \in \mathbb{R}$. Notice that

$-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$. Therefore

by adding these inequalities we get

$$-(|x|+|y|) \leq x+y \leq |x|+|y|.$$

By Fact 5 in the notes above, this implies

$$|x+y| \leq |x|+|y|.$$