Math 206 — Order in the reals, part II

Problem 1. Let $S = \mathbb{Q} \cap (0, 4)$. Consider the following incorrect proof that $\sup S = 4$. What is the logical flaw with this proof?

Proof. First we show 4 is an upper bound. Let $x \in S$. Then $0 < x < 4$, so 4 is clearly an upper bound. Suppose to the contrary that 4 is not the supremum. Then there exists an upper bound u with $u < 4$. But $u < (u + 4)/2$ and we have shown that u is not an upper bound of S. This shows that 4 must be the supremum. □

Problem 2. Fix the proof above, taking advantage of the fact that between any two real numbers there is a rational number.

Problem 3. Let $S = \{1 - 1/n : n \in \mathbb{Z}^+\}$. Show that $\sup S = 1$.