Math 206 — Inequalities and proofs by cases

Problem 1. Our goal in this problem will be to prove the triangle inequality, which is a theorem that says: if $x, y \in \mathbb{R}$ then $|x + y| \leq |x| + |y|$. Start by proving each of the following cases, which break things down into assumptions about x and y.

a. Prove the case when $x > 0$ and $y > 0$.

b. Prove the case when $x < 0$ and $y < 0$.

c. Consider the case when $x < 0 < y$.

   1. Prove the subcase when $|x| \leq y$.
   2. Prove the subcase when $|x| > y$.

d. What case is left to do? What can you say about this case?

e. After class, put your work together into a full proof.

Problem 2. Sometimes it is useful to use the old trick of adding zero: for any real numbers x and a we have $x = x + a - a$. Use this trick along with the triangle inequality to prove that if $x \in \mathbb{R}$ and if $|x - 3| \leq 1$ then $|x^2 - 9| \leq 7$.

Problem 3. Let $x \in \mathbb{R}$. Prove that if $|x + 1| \leq 2$ then $|(x + 1)(x^3 - 2)| \leq 58$.

Problem 4. Express the following sets as open or closed intervals using interval notation.

a. $\{x \in \mathbb{R} : |x| < 5\}$

b. $\{x \in \mathbb{R} : |x| \leq 1\}$

c. $\{x \in \mathbb{R} : |x - 1| < 3\}$

d. $\{x \in \mathbb{R} : |x + 2| \leq 8\}$