Math 241, Spring 2022 — Exam 1

Mount Holyoke College

Due March 11 at 5:00 pm

Instructions. This exam consists of 4 questions, with multiple parts, for a total of 50 points. Please do all of them. To receive full credit, you must show your work, including screenshots of graphs made using Desmos or MATLAB, and provide details and justification where appropriate. You may use your class notes, the textbook, any materials posted to the class web page, Desmos, and MATLAB. However, you may not use other resources (eg. other textbooks or sites on the internet other than our class web page), and you should avoid discussing any aspect of the exam with anyone (except me, Tim). Please submit your work on Gradescope.

Note. I know that you've all been working hard, both in this class and outside of it. I want you to be proud of the effort that you've put in, proud of your individual growth so far, and proud of your integrity. Part of that means taking the honor code seriously, and working on this exam by yourself, without any collaboration or help from classmates or outside materials. I write all of this not because I suspect that you'll cheat, but because I want you to know that I value you each as individuals and value your work and ideas, right or wrong.

Problem 1 (10 points)**.** Consider the map $F(x) = x^3 - 2x^2 + x$.

1. Find all fixed points of this map and classify each as attracting, repelling, weakly attracting (from the left, the right, or both sides), or weakly repelling (from the left, right, or both sides). Your justification should use calculus and algebra, and cite any previously established results from class or homework.

2. Sketch cobweb diagrams of orbits to verify and demonstrate the behavior discussed in the previous part. You may do this Desmos or MATLAB.

Problem 2 (20 points)**.** Consider the logistic map $F(x) = 2.5x(1-x)$ in the context of population modeling, where x represents the size of the population relative to its maximum capacity (represented by $x = 1$). Suppose the population being modeled is a species of fish and the parks service is considering implementing a regimen of replenishing the population. In each successive generation they plan to add to the environment a proportion $0 < c < 1$ of the current population size. This regimen yields a new model $F_c(x) = 2.5x(1-x) + cx$.

1. Suppose $c = 0$ so that no replenishment takes place. Give the fixed points of the system and explain with calculus-based and graphical justification the long term behavior of the system given any initial seed $0 < x_0 < 1$.

2. Suppose $0 < c < 1$ so that replenishment takes place. Give the fixed points of the system and explain how c affects the long term behavior of orbits that start near these fixed points. That is, for which values of c are they attracting vs. neutral vs. repelling?

3. Consider the case $c = 0.6$. Use a combination of Desmos and the `cobweb.m` MATLAB script to explain the long-term behavior of non-fixed and non-periodic initial seeds. Your answer should discuss periodic points and you should show the behavior of four initial seeds, chosen to be representative of the general behavior of the system. Explain using Desmos graphs why you chose your initial seeds. Interpret the long behavior in the context of the fish population.

Problem 3 (10 points)**.** Suppose that we are given a map F that has a neutral fixed point p such that $F'(p) = -1$.

1. Show that p is a fixed point for F^2.
2. Use the chain rule to show that p is also a neutral fixed point for F^2.
3. Use the chain rule and the second derivative of F^2 to discuss the concavity of F^2 at p.
4. Based on your work above, explain which of the following behaviors of orbits starting near p under iteration of F^2 are possible:
   (a) weakly attracting toward p from the left, weakly repelling away from p from the right
   (b) weakly repelling away from p from the left, weakly attracting toward p from the right
   (c) weakly attracting toward p from both sides
   (d) weakly repelling away from p from both sides

Problem 4 (10 points)**.** True or false: for each statement below, either give a brief justification for why it's true, or explain why it's false with a short explanation or counterexample.

1. If p is a neutral fixed point of F^2 then it is a neutral fixed point of F.
2. If F is a quadratic polynomial with 2 fixed points, then it can have at most 4 period-2 points that are not fixed points.
3. Suppose p_1, p_2, p_3 form a 3-cycle of F and $F'(p_1), F'(p_2), F'(p_3) < 1$. Then the 3-cycle must be attracting.