Math 241, Spring 2022 — Exam 2

Mount Holyoke College

Due May 9 at noon

Instructions. This exam consists of 4 questions, with multiple parts, for a total of 50 points. Please do all of them. To receive full credit, you must show your work, including screenshots of graphs made using Desmos or MATLAB, and provide details and justification where appropriate. You may use your class notes, the textbook, any materials posted to the class web page, Desmos, and MATLAB. However, you may not use other resources (eg. other textbooks or sites on the internet other than our class web page), and you should avoid discussing any aspect of the exam with anyone (except me, Tim). Please submit your work on Gradescope.

Note. I know that you've all been working hard, both in this class and outside of it. I want you to be proud of the effort that you've put in, proud of your individual growth so far, and proud of your integrity. Part of that means taking the honor code seriously, and working on this exam by yourself, without any collaboration or help from classmates or outside materials. I write all of this not because I suspect that you'll cheat, but because I want you to know that I value you each as individuals and value your work and ideas, right or wrong.

Problem 1 (20 points)**.** Consider the family of maps given by $F_c(x) = c + (c+1)x + x^2$.

1. Use Desmos to graphically demonstrate that this family undergoes two different saddle-node bifurcations at some values c_0 and c_1. Your solution to this problem should be composed of six graphs generated by Desmos—three for each bifurcation value—and a rough estimate of the values of c_0 and c_1.
2. Use algebra to find the values of c when F_c has 0 fixed points, 1 fixed point, and 2 fixed points. Write a conclusion statement about the exact values of c_0 and c_1.
3. Use Desmos to graphically demonstrate that this family undergoes two different period-doubling bifurcations at some values c_2 and c_3. Your solution to this problem should be composed of six graphs generated by Desmos—three for each bifurcation value—and a rough estimate of the values of c_2 and c_3.
4. Use algebra and calculus to find the exact values of c_2 and c_3.

Problem 2 (15 points)**.** Consider the map

$$T(x) = \begin{cases} 4x & x \leq \frac{1}{2}, \\ 4 - 4x & x > 1/2 \end{cases}$$

and for each $k \geq 1$, let A_k be the set of initial seeds $x_0 \in [0,1]$ that leave $[0,1]$ after k iterations.

1. Use Desmos to graphically demonstrate that $A_1 = (1/4, 3/4)$ and $A_2 = (1/16, 3/16) \cup (13/16, 15/16)$. Your solution should be one or two Desmos graphs and a short explanation of why they serve as a sufficient demonstration.
2. Use Desmos to find A_3. Your solution should be a Desmos graph and the intervals that comprise A_3. Make sure to give the endpoints of A_3 as exact fractions.
3. Find the total length of $\cup_{k=1}^{\infty} A_k$ by extrapolating the pattern of lengths observed for A_1, A_2, A_3.

4. Consider the set $S = [0,1] - \cup_{k=1}^{\infty} A_k$ that results from removing all the sets A_k from the interval $[0,1]$. Is S empty? Explain why or why not.

Problem 3 (10 points). Consider the ternary expansion $x = 0.021\overline{2}$ and the decimal value $y = 1/16$.

1. Find the decimal value represented by x.
2. Is x in the Cantor middle thirds set? Explain why or why not.
3. Find the ternary expansion of y.
4. Is y in the Cantor middle thirds set? Explain why or why not.

Problem 4 (5 points). Pick one topic presented by our classmates during the last week of class and write a short summary of their topic. Explain what you found interesting, why you found it interesting, and any feedback or comments about the presentation or topic. Your answer should be about a page long.